Homework 1

Exercise 1: Particle on a ring, path integral

The Euclidian path integral for a particle on a ring with magnetic flux through the ring is given by

$$Z = \int \mathcal{D}\phi \; e^{-\int_0^\beta d\tau \left(\frac{m\dot{\phi}^2}{2} - i\frac{\theta}{2\pi}\dot{\phi} \right)}.$$

Using the decomposition

$$\phi(\tau) = \frac{2\pi}{\beta}Q\tau + \sum_{l\in\mathbf{Z}} \phi_l e^{i\frac{2\pi}{\beta}l\tau}$$

rewrite the partition function as a sum over topological sectors labeled by winding number $Q \in \mathbf{Z}$ and calculate it explicitly. Find the energy spectrum from the obtained expression.

Hint: Use summation formula

$$\sum_{n=-\infty}^{+\infty} e^{-\frac{1}{2}An^2 + iBn} = \sqrt{\frac{2\pi}{A}} \sum_{l=-\infty}^{+\infty} e^{-\frac{1}{2A}(B-2\pi l)^2}.$$

Exercise 2: spin 1/2 from a particle on a ring

Calculate the partition function of a particle on a ring described in the previous exercise. Find explicit expressions in the limit $M \to 0$, $\theta \to \pi$ but $\theta - \pi \sim M/\beta$. One can interpret the obtained partition function as a partition function of a spin 1/2. What is the physical meaning of the ratio $(\theta - \pi)/M$ in the spin 1/2 interpretation of the result?

Exercise 3: Metric independence of the topological term

The classical action for a particle on a ring is given by

$$S = \int dt_p \left(\frac{m\dot{\phi}^2}{2} - \frac{\theta}{2\pi}\dot{\phi} \right),$$

where t_p is some "proper" time. Reparametrizing time as $t_p = f(t)$ we have $dt_0 = f' dt$ and $dt_0^2 = {f'}^2 dt^2$ and identify the metric as $g_{00} = {f'}^2$ and $g^{00} = {f'}^{-2}$. We also have $\sqrt{g_{00}} = f'$. Rewrite the action in terms of $\phi(t)$ instead of $\phi(t_p)$. Check that it has a proper form if written in terms of the introduced metric. Using the general formula for variation of the action with respect to a metric ($g = \det g_{\mu\nu}$)

$$\delta S = \int dx \sqrt{g}\; T_{\mu\nu}\delta g^{\mu\nu},$$

find the stress-energy tensor for the particle on the ring. Check that T_{00} is, indeed, the energy of the particle.

Exercise 4: Topology of configurational spaces

What is the configuration space of

a) Double spherical pendulum with suspension point which is allowed to move along straight line.

b) Quantum diatomic molecule made out of identical atoms (e.g, N_2). Assume that at the relevant energy scale one can neglect the change of the distance between atoms.

c) Rigid body.

Exercise 5: Classic surfaces

Show that

a) The projective plane RP^2 with a hole is homeomorphic to the Möbius band.

b) Two Klein bottles connected by a pipe are homeomorphic to the Klein bottle with handle.

c) Two projective planes RP^2 connected by a pipe are homeomorphic to the Klein bottle.

Exercise 6: Stereographic projection

Stereographic projection maps a sphere parameterized by a unit vector $\vec{n} = (n_1, n_2, n_3)$ ($\vec{n}^2 = 1$) onto a plane tangent to a sphere at south pole ($\vec{n} = (0, 0, -1)$). The points of the plane are parameterized by Cartesian coordinates (x, y).

a) Find an explicit relation between x, y and $\vec{n}$.

b) The same but use polar coordinates ρ, ϕ instead of x, y.

c) The same but use complex coordinates $w = x + iy$.

Exercise 7: Topological invariant: $S^3 \to S^3$

Consider a three-dimensional unit vector field $\vec{\pi} \in S^3$ on a three-dimensional space $\vec{\pi}(x, y, z)$ with constant boundary conditions $\vec{\pi}(x, y, z) \to (0, 0, 0, 1)$ as $(x, y, z) \to \infty$. Show that

$$Q = A \int d^2x \, \epsilon^{\mu\nu\lambda} \epsilon^{abcd} \pi^a \partial_\mu \pi^b \partial_\nu \pi^c \partial_\lambda \pi^d \tag{1}$$

is an integer-valued topological invariant with properly chosen normalization constant A. Namely,

a) Show that under small variation $\delta\vec{\pi}$ of a vector field the corresponding variation $\delta Q = 0$.

b) Show that the integrand in (1) is a Jacobian of the change of variables from x, y, z to a sphere $\vec{\pi}$ up to normalization.

c) Choose A so that it is normalized in such a way that the area of the 3-sphere is 1. Therefore, Q is an integer degree of mapping of a space (with constant boundary conditions) onto a 3-sphere.

Hint: In b) consider the vicinity of the northern pole of the 3-sphere only and then extend your result to the whole 3-sphere by symmetry.