Homework 2

Problems with stars are not for credit and will NOT be graded.

Exercise 1: Nematic

Nematic is a liquid crystal characterized by an order parameter which is the unit three-component vector $\vec{n} = (n_1, n_2, n_3)$, $\vec{n}^2 = 1$ with an additional condition $\vec{n} \sim -\vec{n}$. The latter means that two unit vectors which are opposite to each other describe the same state.

What types of topological defects and textures are allowed for three-dimensional nematic? What about two-dimensional one?

Exercise 2: Crystal

One can view a crystalline state as continuous translational symmetry broken to the subgroup of discrete translations. Then the order parameter space should be identified (for three-dimensional crystal) with $M = G/H = R^3/(Z \times Z \times Z)$.

a) What (geometrically) is the order parameter space for this system?

b) What are the homotopy groups of this manifold $\pi_{0,1,2,3}(M)$?

c) What types of topological defects and textures are allowed in such a system?

Exercise 3: Superfluid ${}^3He - A$

The order parameter of superfluid ${}^3He - A$ can be represented by two mutually orthogonal unit vectors $\vec{\Delta}_1$, $\vec{\Delta}_2$. That is, at each point in three-dimensional space one has a pair of vectors with properties $\vec{\Delta}_1^2 = \vec{\Delta}_2^2 = 1$ and $\vec{\Delta}_1 \cdot \vec{\Delta}_2 = 0$.

a) What is the manifold of degenerate states for this system?

b) What are the homotopy groups of this manifold $\pi_{0,1,2,3}(M)$?

c) What types of topological defects and textures are allowed in such a system?

*Exercise 4: Heisenberg model

What topological defects and textures one should expect in the ordered state of a three-dimensional classical Heisenberg model? What changes if the order parameter is a director instead of a vector? A "director" means a vector without an arrow, i.e., one should identify $\vec{S} \equiv -\vec{S}$. The models with a director as an order parameter are used to describe nematic liquid crystals.

Exercise 5: Topological invariant: $S^2 \to S^2$

Consider a three-dimensional unit vector field $\vec{n} \in S^2$ on a two-dimensional plane $\vec{n}(x, y)$ with constant boundary conditions $\vec{n}(x, y) \to \hat{e}_3$ as $(x, y) \to \infty$. Show that

$$Q = \int d^2x \, \frac{1}{8\pi} \epsilon^{\mu\nu} \vec{n}[\partial_\mu \vec{n} \times \partial_\nu \vec{n}] \tag{1}$$

is an integer-valued topological invariant. Namely,

a) Show that under small variation $\delta\vec{n}$ of a vector field the corresponding variation $\delta Q = 0$.

b) Show that the integrand in (3) is a Jacobian of the change of variables from x, y to a sphere $\vec{n}$ and it is normalized in such a way that the area of the sphere is 1. Therefore, Q is an integer degree of mapping of a plane (with constant boundary conditions) onto a sphere.

Hint: In b) consider the vicinity of the northern pole of the sphere only and extend your result to the whole sphere by symmetry.

Exercise 6: Bogomol'nyi inequality

Consider the "action" of a two-dimensional O(3) non-linear sigma model

$$S = \frac{1}{2g}\int d^2x\,(\partial_\mu \vec{n})^2. \tag{2}$$

Find the lower bound of this action in a topological sector specified by an invariant Q

$$Q = \int d^2x\,\frac{1}{8\pi}\epsilon^{\mu\nu}\vec{n}[\partial_\mu\vec{n}\times\partial_\nu\vec{n}]. \tag{3}$$

Namely, consider an obvious inequality

$$\int d^2x\,\left(\partial_\mu\vec{n}\pm\epsilon^{\mu\nu}[\vec{n}\times\partial_\nu\vec{n}]\right)^2 \geq 0, \tag{4}$$

open the square and derive an inequality on S in Q sector.

*Exercise 7: Belavin-Polyakov instantons

Let us show that the lower bound found in the previous problem can be achieved. Namely, consider the "self-dual" equation

$$\partial_\mu\vec{n} = -\epsilon^{\mu\nu}[\vec{n}\times\partial_\nu\vec{n}]. \tag{5}$$

We are going to solve this equation in a topological sector Q. Introduce complex coordinates $z = x + iy$, $\bar{z} = x - iy$ and replace $\vec{n}$ by a complex field w (stereographic projection)

$$n_1 + in_2 = \frac{2w}{1+|w|^2}, \tag{6}$$

$$n_3 = \frac{1-|w|^2}{1+|w|^2}. \tag{7}$$

a) Write down an equation (5) in terms of w and z. What is its the most general solution?

b) Derive an expression for Q in terms of w. What is the value of Q in terms of numbers of zeros and poles of w?

c) Write down the most general solution of (5) in terms of w for constant boundary conditions and in a topological sector Q.

References

[1] N. D. Mermin, Rev. Mod. Phys. **51**, 591 (1979). *The topological theory of defects in ordered media.*

[2] Polyakov, A. M. (1987). *Gauge Fields and Strings* (Harwood Acad. Publ. 1987).