Homework 3

Problems with stars are not for credit and will NOT be graded.

*Exercise 1: Continuum limit of XY model

Let us start with the XY model defined on a cubic d-dimensional lattice. The allowed configurations are parameterized by a planar unit vector $\vec{n}_i = (\cos\theta_i, \sin\theta_i)$ on each site i of the latttice. The energy is given by

$$E = -\sum_{\langle ij\rangle} J\cos(\theta_i - \theta_j). \tag{1}$$

We assume that the most important configurations are smooth on a lattice scale and one can think of θ_i as of smooth function $\theta(\vec{x})$ defined in R^d - continuous d-dimensional space. Show that the energy is given in this continuous limit by

$$E = \frac{J}{2}\int \frac{d^d x}{a^d}\, a^2(\partial_\mu\theta)^2, \tag{2}$$

where a is the lattice constant. The combination $\rho_s^{(0)} = Ja^{2-d}$ is referred to as *bare spin-wave stiffness* (or *bare superfluid density*).

Compute the energy of the vortex in such a model (for $d = 2$). Remember that the divergent integrals should be cut by lattice constant a and by the size of the system L at small and large distances respectively.

Exercise 2: Correlation functions

a) Calculate the correlation function $\langle(\theta(x) - \theta(0))^2\rangle$ in the XY model in d dimensions neglecting the topology of θ, i.e., neglect vortices and think about θ as of real number without periodicity. Divergencies at small distances should be cut off by the lattice constant a.

b) Using the result calculate the correlation function $\langle\vec{n}(x)\vec{n}(0)\rangle$ in the XY model in d dimensions neglecting the topology of θ. Write $\langle\vec{n}(x)\vec{n}(0)\rangle = \langle\cos(\theta(x) - \theta(0))\rangle = \mathrm{Re}\,\langle e^{i(\theta(x)-\theta(0))}\rangle$ and use the properties of Gaussian integrals.

c) Make a conclusion about the existence of a true long range order in XY model in 2d and relate it to Mermin-Wagner theorem.

*Exercise 3: Correlation function $\langle\vec{n}(x)\vec{n}(0)\rangle$ at high temperature

Let us now start with high temperatures. Assume that $J/T \ll 1$. Using high temperature expansion show that correlation function $\langle\vec{n}(x)\vec{n}(0)\rangle$ decays exponentially. Find the correlation length at high temperatures.

Exercise 4: Vortex unbinding

Make an estimate of the BKT phase transition temperature in 2d XY model. Use the energy of the vortex calculated previously, the estimate of the entropy of the vortex, and the condition $F = 0$ for the free energy of the vortex.

Exercise 5: WZW in $0+1$, preliminaries

Consider a three-dimensional unit vector field $\vec{n}(x,y)$ ($\vec{n} \in S^2$) defined on a two-dimensional disk D. Define

$$W_0 = \int_D d^2x \, \frac{1}{8\pi} \epsilon^{\mu\nu} \vec{n}[\partial_\mu \vec{n} \times \partial_\nu \vec{n}] = \int_D \frac{1}{16\pi i} \mathrm{tr}\,[\hat{n} d\hat{n} d\hat{n}], \tag{3}$$

where the latter expression is written in terms of differential forms and $\hat{n} = \vec{n} \cdot \vec{\sigma}$.

a) Calculate the variation of W_0 with respect to $\vec{n}$. Show that the integral becomes the integral over disk D of the complete divergence (of the exact form).

b) Parametrize the boundary ∂D of the disk by parameter t, apply Gauss-Stokes theorem and express the result of the variation using only the values of $\vec{n}(t)$ at the boundary.

We showed that the variation of W depends only on the boundary values of $\vec{n}$-field.

Exercise 6: WZW in $0+1$, definition

Assume that we are given the time evolution of $\vec{n}(t)$ field ($\vec{n} \in S^2$). We also assume that time can be compactified, i.e. $\vec{n}(t=\beta) = \vec{n}(t=0)$. Consider the two-dimensional disk D which boundary ∂D is parametrized by time $t \in [0, \beta]$. The WZW term is defined by

$$S_{WZW} = i4\pi S \, W_0[\vec{n}], \tag{4}$$

where S is some constant, W_0 is given by Eq. (3), and $\vec{n}(x,y)$ is some arbitrary smooth extension of $\vec{n}(t)$ from the boundary to an interior of the disk.

Let us show that the WZW term is well defined and (almost) does not depend on the extension of $\vec{n}(t)$ to the interior of D.

Consider two different extensions $\vec{n}^{(1)}(x,y)$ and $\vec{n}^{(2)}(x,y)$ of the same $\vec{n}(t)$ and corresponding values $W_0^{(1)}$ and $W_0^{(2)}$ of the functional W_0. Show that the difference $W_0^{(1)} - W^{(2)}$ is an integer number - the degree Q of mapping $S^2 \to S^2$. The second S^2 here is a target space of $\vec{n}$. How did the first S^2 appear?

We see that $S_{WZW}[\vec{n}(t)]$ is a multi-valued functional which depends on the extension of $\vec{n}$ to the disk D. However, the weight in partition function is given by $e^{-S_{WZW}}$ and can be made single-valued functional if the coupling constant S is "quantized". Namely, if $2S \in \mathbf{Z}$ (S - half-integer number) the $e^{-S_{WZW}}$ is a well-defined single-valued functional.

Exercise 7: WZW in $0+1$, spin precession

Let us consider the quantum-mechanical action of the unit vector $\vec{n}(t)$ with the (Euclidian) action

$$S_h = S_{WZW}[\vec{n}(t)] - S \int dt \, \vec{h} \cdot \vec{n}(t), \tag{5}$$

where W is given by (3) and $\vec{h}$ is some constant three-component vector (magnetic field).

Find the classical equation of motion for $\vec{n}(t)$ from the variational principle $\delta S_h = 0$. Remember that one has a constraint $\vec{n}^2 = 1$ which can be taken into account using, e.g., Lagrange multiplier trick.

The obtained expression is the equation of spin precession and S_{WZW} is a proper, explicitly $SU(2)$ invariant action for the free spin S.

*Exercise 8: WZW in $0+1$, quantization

Show that the classical equations of motion obtained from S_h correspond to Heisenberg equations (in real time) $\partial_t \hat{\vec{S}} = i \left[H, \hat{\vec{S}} \right]$ for the quantum spin operator $\hat{\vec{S}}$

$$\left[S^a, S^b \right] = i\epsilon^{abc} S^c \tag{6}$$

obtained from the Hamiltonian of a spin in magnetic field

$$H = -\vec{h} \cdot \hat{\vec{S}}. \tag{7}$$

Obtain the commutation relations of quantum spin (6) from the topological part S_{WZW}. Notice that this topological action is linear in time derivative and, therefore, does not contribute to the Hamiltonian. Nevertheless, it defines commutation relations between components of the spin operator.

Hint: You can either use local coordinate representation of the unit vector in terms of spherical angles $\vec{n} = (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)$ or use the general formalism of obtaining Poisson bracket from the symplectic form given in S_{WZW}.

*Exercise 9: Reduction of WZW to the theta-term in $0+1$

Let us assume that the field $\vec{n}(t)$ is constrained so that it takes values on a circle given in spherical coordinates by $\theta = \theta_0 = const$. Find the value of the topological term S_{WZW} on such configurations (notice that this constraint is not applicable in the interior of the disk D, only at its physical boundary). Show that the obtained topological term is a theta-term in $0+1$ corresponding to $S^1 \to S^1$.

What is the value of the coefficient in front of that topological term? What is the value of corresponding "magnetic flux" through a ring? For $S = 1/2$ which reduction (value of θ_0) corresponds to the half of the flux quantum?

References

[1] N. D. Mermin, Rev. Mod. Phys. **51**, 591 (1979).
The topological theory of defects in ordered media.

[2] Polyakov, A. M. (1987). *Gauge Fields and Strings* (Harwood Acad. Publ. 1987).