Homework 4

Problems with stars are not for credit and will NOT be graded.

Exercise 1: Forms on R^3

Consider the oriented euclidian space R^3 (with a given scalar product). Every vector $\vec{A} \in R^3$ determines a 1-form by $\omega^1_{\vec{A}}(\vec{\xi}) = (\vec{A} \cdot \vec{\xi})$ and a 2-form by $\omega^2_{\vec{A}}(\vec{\xi}_1, \vec{\xi}_2) = (\vec{A} \cdot [\vec{\xi}_1 \times \vec{\xi}_2])$.

Show that

$$a) \qquad \omega^1_{\vec{A}} \wedge \omega^1_{\vec{B}} = \omega^2_{[\vec{A}\times\vec{B}]},$$

$$b) \qquad \omega^1_{\vec{A}} \wedge \omega^2_{\vec{B}} = (\vec{A} \cdot \vec{B}) x_1 \wedge x_2 \wedge x_3.$$

Exercise 2: Vector calculus on 3-dimensional manifold

In a 3-dimensional oriented riemannian space M, every vector field $\vec{A}(x)$ corresponds to a 1-form $\omega^1_{\vec{A}}$ and $\omega^2_{\vec{A}}$ defined similarly to exterior forms in a previous exercise.

a) Show that

$$df = \omega^1_{\nabla f}, \qquad d\omega^1_{\vec{A}} = \omega^2_{\nabla\times\vec{A}}, \qquad d\omega^2_{\vec{A}} = (\nabla \cdot \vec{A})\omega^3, \tag{1}$$

where f is a 0-form (scalar function) and ω^3 is a volume 3-form on M.

b) Identify the Stoke's formula applied to each equation in (1) with known formulae of vector calculus.

Exercise 3: Vector calculus on 3-dimensional manifold

Using differential forms and the results of previous exercises show that

$$\begin{aligned} a) \qquad \nabla \cdot [\vec{A} \times \vec{B}] &= (\nabla \times \vec{A}) \cdot \vec{B} - (\nabla \times \vec{B}) \cdot \vec{A}, \\ b) \qquad \nabla \times (a\vec{A}) &= (\nabla a) \times \vec{A} + a(\nabla \times \vec{A}), \\ c) \qquad \nabla \cdot (a\vec{A}) &= (\nabla a) \cdot \vec{A} + a(\nabla \cdot \vec{A}). \end{aligned} \tag{2}$$

Hint: Use the formula for the derivative of the product of forms.

Exercise 4: Fermionic determinant in two dimensions

Let us consider two-dimensional fermions coupled to a phase field $\phi(x)$ ($\phi \equiv \phi + 2\pi$). The Euclidian Lagrangian is given by

$$\mathcal{L}_2 = \bar{\psi}\left[i\gamma^\mu(\partial_\mu - iA_\mu) + ime^{i\gamma^5\phi}\right]\psi, \tag{3}$$

where $\mu = 1, 2$ is a spacetime index, $\gamma^{1,2,5}$ is a triplet of Pauli matrices, and A_μ is an external gauge field probing fermionic currents.

We assume that the bosonic field ϕ changes slowly on the scale of the "mass" m. Then one can integrate out fermionic degrees of freedom and obtain an induced effective action for the ϕ-field as a functional determinant.

$$\begin{aligned} S_{eff} &= -\log \mathrm{Det}\, D, &(4)\\ D &= i\gamma^\mu(\partial_\mu - iA_\mu) + im\, e^{i\gamma^5\phi}. &(5)\end{aligned}$$

We calculate the effective action using the gradient expansion method. Namely, we calculate the variation of (4) with respect to the ϕ and A-fields and use

$$\delta S_{eff} = -\delta \log \mathrm{Det}\, D = -\mathrm{Tr}\, \delta \log D = -\mathrm{Tr}\, \delta D\, D^{-1} = -\mathrm{Tr}\, \delta D\, D^\dagger (DD^\dagger)^{-1}. \qquad (6)$$

a) Calculate $DD^\dagger$ for (5). Observe that this object depends only on gradients of ϕ-field.

b) Expand $(DD^\dagger)^{-1}$ in those gradients. This will be the expansion in $1/m$. (It is convenient to introduce notation $G_0^{-1} = -\partial_\mu^2 + m^2$).

c) Calculate functional traces of the terms up to the order of m^0. Use the plane wave basis to calculate the trace $\mathrm{Tr}\,(\hat{X}) \to \int d^2x \int \frac{d^2p}{(2\pi)^2} e^{-i\vec{p}\cdot\vec{x}} \hat{X} e^{i\vec{p}\cdot\vec{x}}$.

d) Identify the variation of the topological term in the obtained expression. It contains the antisymmetric tensor $\epsilon^{\mu\nu}$ and is proportional to $\mathrm{sgn}\,(m)$.

e) Remove the variation from the obtained expression and find S_{eff} up to the m^0 order.

f) Which terms of the obtained action are topological? Can you write them in terms of differential forms?

*Exercise 5: "Dangers" of chiral rotation

Try to calculate the determinant of the previous exercise using "chiral rotation trick". Namely, consider chiral rotation $\psi \to e^{-i\gamma^5\phi/2}\psi$. Then $\psi^\dagger \to \psi^\dagger e^{i\gamma^5\phi/2}$ and $\bar\psi \to \bar\psi e^{-i\gamma^5\phi/2}$. Use the identity $\gamma^\mu\gamma^5 = -i\epsilon^{\mu\nu}\gamma^\nu$ and anti-commutativity of Pauli matrices to show that the operator $D(A_\mu, \phi)$ transforms into

$$\tilde{D}(A_\mu, \phi) = e^{-i\gamma^5\phi/2} D(A_\mu, \phi) e^{-i\gamma^5\phi/2} = D(A_\mu + \frac{i}{2}\epsilon^{\mu\nu}\partial_\nu\phi, 0) = D(\tilde{A}_\mu, 0).$$

Try to calculate calculate $\log \mathrm{Det}\, \tilde{D} = \log \mathrm{Det}\, D(\tilde{A}, 0)$ using expansion in $\tilde{A}$. You will see that the result does not match the effective action obtained in the previous exercise. Why? What one should add to the chiral rotation trick to make the correct calculation?

References

[1] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer; 2nd edition (1997).

[2] Mikio Nakahara, *Geometry, Topology, and Physics*, 2nd edition, Taylor & Francis, New York, 2003.

[3] M. Monastyrsky, O. Efimov (Translator), "Topology of Gauge Fields and Condensed Matter" Plenum Pub Corp; March 1993; ISBN: 0306443368

[4] Abanov, A.G. and Wiegmann, P.B. (2000). *Theta-terms in nonlinear sigma-models*, Nucl. Phys. **B 570**, 685-698.