Homework 5

Problems with stars are not for credit and will NOT be graded.

Exercise 1: Haldane's theta-term from WZW in $0+1$

Following Haldane, we consider the chain of quantum spins described by the action

$$S_{chain} = \sum_j i4\pi S\, W_0[\vec{n}_j(t)], \tag{1}$$

where the sum is taken over the sites of a 1d spin chain labeled by j. Let us assume that as a result of the dynamics we have an antiferromagnetic chain and that the field $(-1)^j\vec{n}_j$ is a smooth one and can be approximated by the smooth field $\vec{n}(x,t)$, where $x = ja$ and a is a lattice constant. Show that (1) reduces to the theta term in $2+0$ dimensions corresponding to the mapping $S^2 \to S^2$. What is the value of the coefficient θ in front of that term?

Exercise 2: Boundary spin 1/2 states of a Haldane's chain

Consider the "action" of a two-dimensional O(3) non-linear sigma model with topological term

$$\begin{aligned} S &= S_{NLSM} + S_\theta, & (2)\\ S_{NLSM} &= \frac{1}{2g}\int d^2x\, (\partial_\mu \vec{n})^2, & (3)\\ S_\theta &= i\theta Q, & (4)\\ Q &= \int d^2x\, \frac{1}{8\pi}\epsilon^{\mu\nu}\vec{n}[\partial_\mu\vec{n}\times\partial_\nu\vec{n}]. & (5)\end{aligned}$$

This action can be derived as a continuum limit of Heisenberg spin chain with large spins on sites. In the latter case $g = 2/S$ and $\theta = 2\pi S$. In the case of integer S the spin chain is massive and there are no bulk excitations at low energies (smaller than the gap).

Let us assume that the action (2) is defined on the open chain of the length L. Show that the topological theta-term formally defined on the open chain reduces to two WZW (0+1-dimensional) terms at the boundary of spacetime, i.e. at the ends of the spin chain. This means that we expect two quantum spins living at the ends of the spin chain. Show that the coefficient in front corresponds to the value of those spins $S/2$. In particular, it means that the boundary states of $S=1$ spin chain correspond to spin-1/2!

Remark: Neglecting the NLSM part of the action is possible in this exercise only because of the gap in the bulk at the integer value of spin.

Exercise 3: Topological current term from the fermionic determinant in three dimensions

Let us consider three-dimensional fermions coupled to NLSM given in terms of the three component unit vector $\vec{n} \in S^3$ (i.e., $n_1^2+n_2^2+n_3^2 = 1$). The Euclidian Lagrangian is given by

$$\mathcal{L}_3 = \bar{\psi}\left[i\gamma^\mu(\partial_\mu - iA_\mu) + im\vec{n}\cdot\vec{\tau}\right]\psi, \tag{6}$$

where $\mu = 1, 2, 3$ is a spacetime index, $\vec{\tau}$ is a vector of Pauli matrices, and A_μ is an external gauge field probing fermionic currents.

We assume that the bosonic field π changes slowly on the scale of the "mass" m. Then one can integrate out fermionic degrees of freedom and obtain an induced effective action for the π-field as a functional determinant.

$$S_{eff} = -\log \mathrm{Det}\, D, \tag{7}$$
$$D = i\gamma^\mu(\partial_\mu - iA_\mu) + im\vec{n}\cdot\vec{\tau}. \tag{8}$$

Calculate the variation of (7) with respect to the external gauge field A_μ up to the order of m^0 and derive the expression for the topological current. What is the (physical) geometrical meaning of the topological current term?

Exercise 4: Topological current for $O(3)$ sigma model in 3 dimensions

The $O(3)$ sigma model is written in terms of the unit vector field $\vec{n}(x)$ ($\vec{n} \in S^2$, or $\vec{n}^2 = 1$). In three (3+0) dimensions one can form a "topological current"

$$j^\mu = \frac{1}{8\pi}\epsilon^{\mu\nu\lambda}\vec{n}\cdot\partial_\nu\vec{n}\times\partial_\lambda\vec{n} \tag{9}$$
$$= \frac{1}{8\pi}\epsilon^{\mu\nu\lambda}\epsilon_{abc}n^a\partial_\nu n^b\partial_\lambda n^c. \tag{10}$$

Show that for smooth fields $\vec{n}(x)$ we have regardless of equations of motion

$$\partial_\mu j^\mu = 0. \tag{11}$$

What is the meaning of the "conserved topological charge"? (Consider $\int_{t=0} d^2x\, j^0$).

Exercise 5: CP^1 representation of $\vec{n}$-field

The unit vector $\vec{n} \in S^2$ can be represented in terms of complex spinor $z = (z_1, z_2)^T$ as

$$\vec{n} = z^\dagger\vec{\sigma}z \tag{12}$$

or alternatively in the matrix form $\hat{n} \equiv \vec{n}\cdot\vec{\sigma} = 2zz^\dagger - \hat{1}$. The condition $\vec{n}^2 = 1$ requires $z^\dagger z = |z_1|^2 + |z_2|^2 = 1$, i.e., $z \in S^3$. This representation is excessive as $z \to e^{i\alpha}z$ does not change $\vec{n}$. Identifying all points of orbits $e^{i\alpha}z \equiv z$ we have the construction $\vec{n} \in S^2 = S^3/S^1$ with $z \in S^3$ and explicit formula (12). This construction is known as CP^1-representation of an $O(3)$ n-field or as Hopf fibration.

Using (12) express the topological current (9) in terms of z and $z^\dagger$. Express it also in terms of the "gauge field" $a_\mu \equiv z^\dagger i\partial_\mu z$. Notice, that under transformation $z(x) \to e^{i\alpha(x)}$ the gauge field transforms properly as $a_\mu \to a_\mu - \partial_\mu\alpha$. Also notice that the expression for the topological current is gauge invariant (as it is supposed to be as a function of $\vec{n}$ only).

*Exercise 6: $S^3 \to S^3$ mappings

The mappings of $S^3 \to S^3$ can be divided into homotopy classes labeled by an integer winding number n according to $\pi_3(S^3) = \mathbf{Z}$. In terms of $SU(2)$ matrix valued function $g(x)$ ($SU(2) \sim S^3$) one can write the winding number as

$$n = N\int_{S^3} d^3x\, \epsilon^{\mu\nu\lambda}\,\mathrm{tr}\,\left[(g^{-1}\partial_\mu g)(g^{-1}\partial_\nu g)(g^{-1}\partial_\lambda g)\right]. \tag{13}$$

Fix the normalization constant N to have any integer number as a possible value for n.

One can parametrize an $SU(2)$ matrix explicitly as $g = \phi^0 + i\vec{\sigma}\cdot\vec{\phi}$ with $(\phi^0)^2 + \vec{\phi}^2 = 1$, i.e., $(\phi^0, \vec{\phi}) \in S^3$. Here $\vec{\sigma}$ is a vector of Pauli matrices. Show that the winding number can also be expressed as ($a, b, c, d = 0, 1, 2, 3$; $\mu, \nu, \lambda = 1, 2, 3$)

$$n = \tilde{N} \int_{S^3} d^3x\, \epsilon^{\mu\nu\lambda} \epsilon_{abcd} \phi^a \partial_\mu \phi^b \partial_\nu \phi^c \partial_\lambda \phi^d. \tag{14}$$

Fix the normalization constant $\tilde{N}$.

*Exercise 7: Pure gauge

Introducing the $SU(2)$ Yang-Mills field (pure gauge)

$$\hat{a}_\mu = a^a_\mu \sigma^a = g^{-1} i \partial_\mu g, \tag{15}$$

derive "zero curvature condition" considering $\partial_\mu \hat{a}_\nu - \partial_\nu \hat{a}_\mu$ and using $\partial_\mu g^{-1} = -g^{-1}\partial_\mu g g^{-1}$.

Using the obtained formulas express (13) in terms of the third component of the gauge field a^3_μ only. Using the parametrization of an $SU(2)$ matrix in terms of $z \in S^3$ as

$$g = \begin{pmatrix} z_1 & -z_2^* \\ z_2 & z_1^* \end{pmatrix}, \tag{16}$$

show that

$$a^3_\mu = z^\dagger i \partial_\mu z, \tag{17}$$

i.e. it is the same "gauge field" that we used for CP^1 representation of the n-field. Show that the result obtained for winding number n is gauge invariant with respect to $z \to e^{i\alpha} z$ and, therefore, is the function of the field $\vec{n} = z^\dagger \vec{\sigma} z$ only.

What is the geometrical meaning of the "winding number" n given by (13) for the $\vec{n}$-field? This is a relatively difficult question.

Exercise 8: Topological current and singularities of $\vec{n}$-field

Let us assume that the $\vec{n}$-field is smooth everywhere on the surface ∂D of the domain D of three-dimensional spacetime. We write down the flux of the topological current through the closed surface ∂D of the domain D.

$$\Phi = \int_{\partial D} j^\mu \, dS_\mu = \int_D \partial_\mu j^\mu \, d^3x. \tag{18}$$

Here $d\vec{S}$ is an area vector directed outwards normal to the surface.

If the $\vec{n}$ field is smooth everywhere inside D the obtained flux is zero because of the topological current conservation. This is not so if there are singularities (defects) of $\vec{n}(x)$ inside the domain D. Explain the geometrical meaning of the flux Φ in this case. Express the flux Φ in terms of singularities inside D. What type of singularities contribute to the flux?

Express Φ in terms of the gauge field a_μ. What is the meaning of Φ in terms of that gauge field? How do singularities inside D look in terms of the gauge field?

References

[1] E. Fradkin, "Field Theories of Condensed Matter Systems", Addison Wesley Longman, Inc.; March 1998; ISBN: 0201328593

[2] Abanov, A.G. and Wiegmann, P.B. (2000). *Theta-terms in nonlinear sigma-models*, Nucl. Phys. **B 570**, 685-698.