Homework 6 (not for credit)

Exercise 1: WZW in $0+1$, derivation from fermions

Consider an Euclidian action of a fermion coupled to a unit vector

$$S_E = \int d\tau\, \psi^\dagger D\psi, \tag{1}$$

where

$$D = \partial_\tau - m\vec{n}\cdot\vec{\tau} \tag{2}$$

with $\vec{n} \in S^2$ and $\vec{\tau}$ the vector of Pauli matrices. We obtain an effective action for $\vec{n}$ induced by fermions as

$$e^{-S_{eff}} = \int D\psi D\psi^\dagger e^{-S_E} = \mathrm{Det}\, D \tag{3}$$

or

$$S_{eff} = -\log \mathrm{Det}\, D = -\mathrm{Tr}\, \log D. \tag{4}$$

We calculate the variation of S_{eff} with respect to $\vec{n}$ as

$$\delta S_{eff} \quad = \quad -\mathrm{Tr}\, \delta D D^{-1} = -\mathrm{Tr}\, \delta D D^\dagger (DD^\dagger)^{-1}, \tag{5}$$

where $D^\dagger = -\partial_\tau - m\vec{n}\cdot\vec{\tau}$. We have

$$DD^\dagger \quad = \quad -\partial_\tau^2 + m^2 - m\dot{\vec{n}}\cdot\vec{\tau} = G_0^{-1} - m\dot{\vec{n}}\cdot\vec{\tau}. \tag{6}$$

Expand (5) in $1/m$ up to the term m^0 and calculate functional traces. Show that the term of the order m^0 is a variation of the WZW term in 0+1 dimensions. Restore S_{eff} from its variation. What is the coefficient in front of the WZW term? To what value of spin does it correspond?

Exercise 2: Fractional charge on solitons (topological current)

Consider the following 1+1 dimensional action

$$S = \int d^2x\, \bar{\psi}\Big(i\not{\partial} + \not{A} + me^{i\phi\gamma^5}\Big)\psi. \tag{7}$$

Here m is some constant and $\phi(x,t)$ is a field. Integrate out fermions (assuming that m is large) and obtain the expression for the fermionic current induced by the non-uniform field $\phi(x,t)$. What is the charge induced by the static configuration $\phi(x)$ with asymptotics $\phi(x\to\pm\infty) = \phi_\pm$?

Exercise 3: Fractional charge on solitons (Peierls transition)

The following continuum model is obtained for a system exhibiting Peierls period doubling transition in 1d molecules.

$$S = \int d^2x\, \bar{\psi}\Big(i\not{\partial} + \not{A} + \Delta(x)\Big)\psi + S[\Delta]. \tag{8}$$

Here $\Delta(x)$ is the order parameter. The lowest energy of the system is achieved when the order parameter is uniform $\Delta(x) = \pm\Delta_0$. The configuration of the order parameter such that $\Delta(x\to\pm\infty) = \pm\Delta_0$ is a topological soliton (corresponding to $\pi_0(Z_2) = Z_2$) a.k.a kink or domain wall. Find the charge induced by this configuration. Prove that the spectrum of the fermionic Hamiltonian corresponding to the action (8) has a zero mode.

Exercise 4: WZW term from the fermionic determinant in two dimensions

Let us consider two-dimensional fermions coupled to NLSM given in terms of the four component unit vector $(\pi_0, \vec{\pi}) \in S^3$ (i.e., $\pi_0^2 + \pi_1^2 + \pi_2^2 + \pi_3^2 = 1$). The Euclidian Lagrangian is given by

$$\mathcal{L}_2 = \bar{\psi}\left[i\gamma^\mu(\partial_\mu - iA_\mu) + im(\pi_0 + i\gamma^5\vec{\pi}\cdot\vec{\tau})\right]\psi, \tag{9}$$

where $\mu = 1, 2$ is a spacetime index, $\vec{\tau}$ is a vector of Pauli matrices, and A_μ is an external gauge field probing fermionic currents.

We assume that the bosonic field π changes slowly on the scale of the "mass" m. Then one can integrate out fermionic degrees of freedom and obtain an induced effective action for the π-field as a functional determinant.

$$S_{eff} = -\log \mathrm{Det}\, D, \tag{10}$$

$$D = i\gamma^\mu(\partial_\mu - iA_\mu) + im(\pi_0 + i\gamma^5\vec{\pi}\cdot\vec{\tau}). \tag{11}$$

We calculate the effective action using the gradient expansion method. Namely, we calculate the variation of (10) with respect to the π-field and use

$$\delta S_{eff} = -\delta \log \mathrm{Det}\, D = -\mathrm{Tr}\,\delta \log D = -\mathrm{Tr}\,\delta D\, D^{-1} = -\mathrm{Tr}\,\delta D\, D^\dagger (DD^\dagger)^{-1}. \tag{12}$$

a) Calculate $DD^\dagger$ for (11). Observe that this object depends only on gradients of π-field.

b) Expand $(DD^\dagger)^{-1}$ in those gradients. This will be the expansion in $1/m$. (It is convenient to introduce notation $G_0^{-1} = -\partial_\mu^2 + m^2$).

c) Calculate functional traces of the terms up to the order of m^0. Use the plane wave basis to calculate the trace $\mathrm{Tr}\,(\hat{X}) \to \int d^2x \int \frac{d^2p}{(2\pi)^2} e^{-i\vec{p}\cdot\vec{x}}\hat{X}e^{i\vec{p}\cdot\vec{x}}$.

d) Identify the variation of the topological term in the obtained expression. It contains the antisymmetric tensor $\epsilon^{\mu\nu}$ and is proportional to $\mathrm{sgn}\,(m)$.

e) Remove the variation from the obtained expression and find S_{eff} up to the m^0 order. Remember that the removal of the variation for topological term requires Wess-Zumino trick (introduce an auxiliary ball D^3 with the spacetime being the boundary of the ball, extend fields to the ball, etc...).

Exercise 5: Parity anomaly

Consider massive 2+1 dimensional Dirac fermions coupled to the gauge field

$$S = \int d^3x\, \bar{\psi}(i\partial\!\!\!/ + A\!\!\!/ + m)\psi. \tag{13}$$

Integrate out fermions and obtain the effective action $S_{eff}[A]$ induced by these fermions in the limit of small field gradients (compared to m). Truncate the calculation at the order $1/m$. What is the topological part of the effective action? This calculation is related to so-called "parity anomaly" [3].

Exercise 6: Aharonov-Casher theorem

Consider the 2-dimensional massless Dirac Hamiltonian in external magnetic field

$$H = \alpha^\mu(i\partial_\mu + A_\mu), \tag{14}$$

where $\mu = x, y$ and α-matrices can be taken as Pauli matrices $\alpha^\mu = \sigma^\mu$. Assume that the total flux of magnetic field (magnetic field is non-uniform) through the plane is $\Phi = \int d^2x \epsilon^{\mu\nu}\partial_\mu A_\nu$. Show that (14) has $[\Phi/\Phi_0] - 1$ normalized zero modes (solutions of $H\psi = 0$). This example of Atiyah-Singer index theorem is known as Aharonov-Casher theorem [4].

Exercise 7: Theta term by reduction from WZW term

Consider the two-dimensional WZW term defined as an integral over three-dimensional ball D^3

$$S_{WZW} = ik\frac{1}{12\pi}\int_{D^3} d^3x\, \epsilon^{\mu\nu\lambda}\,\mathrm{tr}\,\left[(g^{-1}\partial_\mu g)(g^{-1}\partial_\nu g)(g^{-1}\partial_\lambda g)\right], \tag{15}$$

where $g \in SU(2)$ is a matrix-valued field. This field is defined on the two-dimensional spacetime $S^2 = \partial D^3$ and smoothly extended to the interior of the ball from the boundary (actual spacetime).

a) Show that (15) is well defined if the coupling constant k is integer. Namely, its value does not depend on the way the field g is extended from S^2 to D^3.

b) Let us make a reduction and substitute in (15) $g = \cos\alpha + i\sin\alpha\,\vec{n}\cdot\vec{\sigma}$ with $\alpha = const$ and $\vec{n} \in S^2$ a unit vector. Show by explicit calculation that (15) reduces to the theta-term made out of $\vec{n}$ and given by the integral over spacetime S^2. Find the corresponding value of θ coupling in terms of α and k.

Exercise 8: Linear response for QHE

The linear response for QHE state can be summarized in the form of Chern-Simons action

$$S_{CS} = \int d^2x\, dt\, \frac{\sigma_{xy}}{4\pi}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda, \tag{16}$$

where a_μ is the deviation of an e/m potential from its background value corresponding to the uniform magnetic field. The linear response is obtained by taking variations of this effective action $j^a = \delta S_{CS}/\delta A_a$ and $\delta\rho = \delta S_{CS}/\delta A_0$.

a) Show that the coefficient σ_{xy} in (16) has a meaning of a Hall conductance and that the longitudinal conductance given by (16) is zero.

b) Using (16) relate $\delta\rho$ with external magnetic field (i.e., obtain the Streda formula). Increasing magnetic field increases the number of filled states - phenomenon known as a spectral flow.

c) Show that if QHE sample is not infinite but has a boundary, the action (16) is not gauge invariant. The latter means that it should be complimented by some boundary action - the action of edge states.

Exercise 9: Edge states for IQHE

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This problem is a toy problem illustrating some physics of an Integer Quantum Hall Effect (IQHE).

Consider the electron gas confined to the two dimensional xy plane. Let us neglect the interaction between electrons. The Hamiltonian of a single particle is given by

$$H = -\frac{1}{2m}\left(-i\hbar\vec{\nabla} + \frac{e}{c}\vec{A}\right)^2 + V(x,y),$$

where $\vec{A}$ is a vector potential of magnetic field and $V(x,y)$ is an additional electrostatic (confining) potential. For simplicity we will take the confining potential to be a one-dimensional harmonic potential $V = \frac{1}{2}m\omega_0^2 y^2$.

a. For the constant magnetic field B using the Landau gauge $A_x = -By$, $A_y = 0$ and a separation of variables $\psi(x,y) = \psi_k(y)e^{ikx}$ write down the stationary Schroedinger equation for $\psi_k(y)$.

b. Identify the obtained equation as the one for harmonic oscillator and find the energy levels $E_{k,n}$ with $n = 0, 1, 2, \ldots$. The levels at given n are said to belong to the same *Landau level*.

c. Let us assume that the chemical potential μ is such that Landau levels with $n > 0$ are empty (i.e., $E_{k,n} > \mu$ for $n > 0$). Then the only occupied states are the ones with $n = 0$. What are the maximal and minimal values of k of occupied levels?

d. What are positions (in y direction) of those occupied levels?

e. The states with maximal and minimal k are called the edge states of IQHE. Find the velocity of corresponding boundary excitations.

Exercise 10: Hall conductance from edge states for IQHE

a) Consider a ballistic wire with $M = 1$ channel. This means that there is 1 right and 1 left moving electronic modes with velocities $\pm v_F$ respectively. Assume that the chemical potential at the right (left) end of the wire is μ_L (μ_R) respectively. Show that in the absence of scattering (ballistic wire) the current in the wire is given by

$$I = \frac{e}{h}(\mu_R - \mu_L)M, \tag{17}$$

where M is the number of channels.

b) In QHE the current is carried by edge modes. Thinking of one side of the Hall bar as of right moving states and of the other one as of left moving states find the Hall conductance of the Hall bar. Show that it is given by

$$\sigma_{xy} = \frac{e^2}{h}M. \tag{18}$$

Hint: one should think of $\mu_R - \mu_L$ as of eV_H - the voltage across the sample.

References

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