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Introduction

1.1 Topology in condensed matter physics as an emergent phenomenon

1.1.1 Spontaneous symmetry breaking and an emergent topology

If there were no separation of scales in Nature the task of theoretical physicist would be formidable. In fact, in many cases one can effectively describe some properties of microscopic systems at low temperatures, low frequencies, and large distances using relatively simple continous field theory descriptions. This happens due to the presence of exact or approximate symmetries in the underlying microscopic system. More precisely, it is due to a phenomenon of spontaneous symmetry breaking.

Suppose that the exact Hamiltonian of some condensed matter system has some continous symmetry given by Lie group G. A good system to keep in mind as an example is some isotropic ferro- or antiferromagnet with an SU(2) symmetry with respect to global rotations of all spins. Then it is possible that at some values of parameters of the Hamiltonian† the ground state of the system breaks the symmetry up to some subgroup H of G. If this happens we say that the symmetry of the Hamiltonian is spontaneously broken by its ground state. One can characterize this ground state by some element n of a coset space G/H. In our example of the magnet we take $H = SO(2) = U(1)$ and $G/H = SU(2)/U(1) = S^2$. The element of a coset space in this case is a point of two-dimensional sphere S^2 which labels the direction of magnetization of our system and subgroup H is just the group of all rotations around the direction of magnetization which is obviously a symmetry of the Hamiltonian and of the ground state of the system. In the presence of spontaneous breaking of a continous symmetry the ground state is infinitely degenerate since any $n \in G/H$ gives the ground state with the

† We consider here the case of zero temperature for simplicity.

same energy. Indeed, any two states characterized $n_1, n_2 \in G/H$ have the same energy since they can be transformed one into another by some element $g \in G$ which is an exact symmetry transformation of the Hamiltonian.

Now consider another state of the quantum system which is locally, in the vicinity of spacial point x is very close to the ground state of the system labeled by some $n(x) \in G/H$. We assume that $n(x)$ is not constant in space but changes very slowly with a typical wavenumber k. $n(x)$ is called an order parameter of the system. We denote the energy of this state per unit volume measured from the energy of the ground state $\epsilon(k)$. The limit of small $k \to 0$ corresponds to the order parameter which is constant in space $n(x) = n_0$ and obviously $\epsilon(k) \to 0$ as $k \to 0$. We obtain that when continous symmetry is spontaneously broken, the ground state of the system is not isolated but there are always excited states which energies are infinitezimally close to the ground state energy. These heuristic arguments can be made more rigorous and lead to a Goldstone theorem†. The theorem states that in quantum field theory with spontaneously broken continuous symmetry there are massless particles which energy $\epsilon(k) \to 0$ as $k \to 0$.

If one is interested in low energy physics one necessarily should take these massless modes (or Goldstone bosons) into account. Moreover, the nature of these massless modes is dictated essentially by the symmetry (and its breaking) of the system and one expects, therefore, that the correct low energy description should depend only on symmetries of the system but not on its every microscopic detail.

A natural variable describing the dynamics of Goldstone modes is the order parameter itself. For example, in the case of relativistically invariant system described by the order parameter $\vec{n} \in S^2$ we immediately write

$$S_{\text{NLSM}} = \int d^{d+1}x \, \frac{1}{2g} (\partial_\mu \vec{n})^2 + (\text{other terms}). \tag{1.1}$$

We have written here the most obvious term of an effective action which is both Lorentz invariant and SU(2) invariant (with respect to rotations of a unit, three-component vector $\vec{n}^2 = 1$). Here g is a coupling constant which should be obtained from a detailed microscopic theory. The "other terms" are the terms which are higher order in gradients and (possibly) topological terms. The model (1.1) is referred to as a "non-linear σ-model"‡.

† For a full formulation of the Goldstone theorem for relativistic field theory as well as for its proof see e.g., [**?**]. We avoid it here because we are generally interested in a wider range of systems, e.g., without Lorentz invariance. There are still some analogs of Goldstone theorem there. For example in the case of a ferromagnet there are still massless particles – magnons. However, the number of independent massless particles is not correctly given by a Goldstone theorem for relativistic systems.

‡ The origin of the term is in effective theories of weak interactions[**?**]. Non-linear comes from

In different spatial dimensions higher gradient terms of non-linear σ-models might be relevant. We are not discussing those terms as well as the issue of renormalizability of our σ-models concentrating instead on the allowed topological terms. Therefore, we will keep only the kinetic term $\frac{1}{2g}(\partial_\mu \vec{n})^2$ in the gradient expansion of an effective Lagrangian as well as all allowed topological terms.

Before proceeding to our main subject – topological terms, let us make two important remarks. First is that very often (especially in condensed matter systems) the symmetries of the Hamiltonian are approximate and there are terms in the Hamiltonian which explicitly but weakly break the symmetry. This does not invalidate the speculations of this section. The difference will be that would-be-Goldstone particles acquire small mass. The weaker an explicit symmetry breaking of the Hamiltonian the smaller the mass of "Goldstone" particles. One can proceed with the derivation of non-linear σ-model which will contain weak symmetry breaking terms (such as easy-axis anisotropy for magnets). This model will have non-trivial dynamics at energies bigger than the smallest of masses.

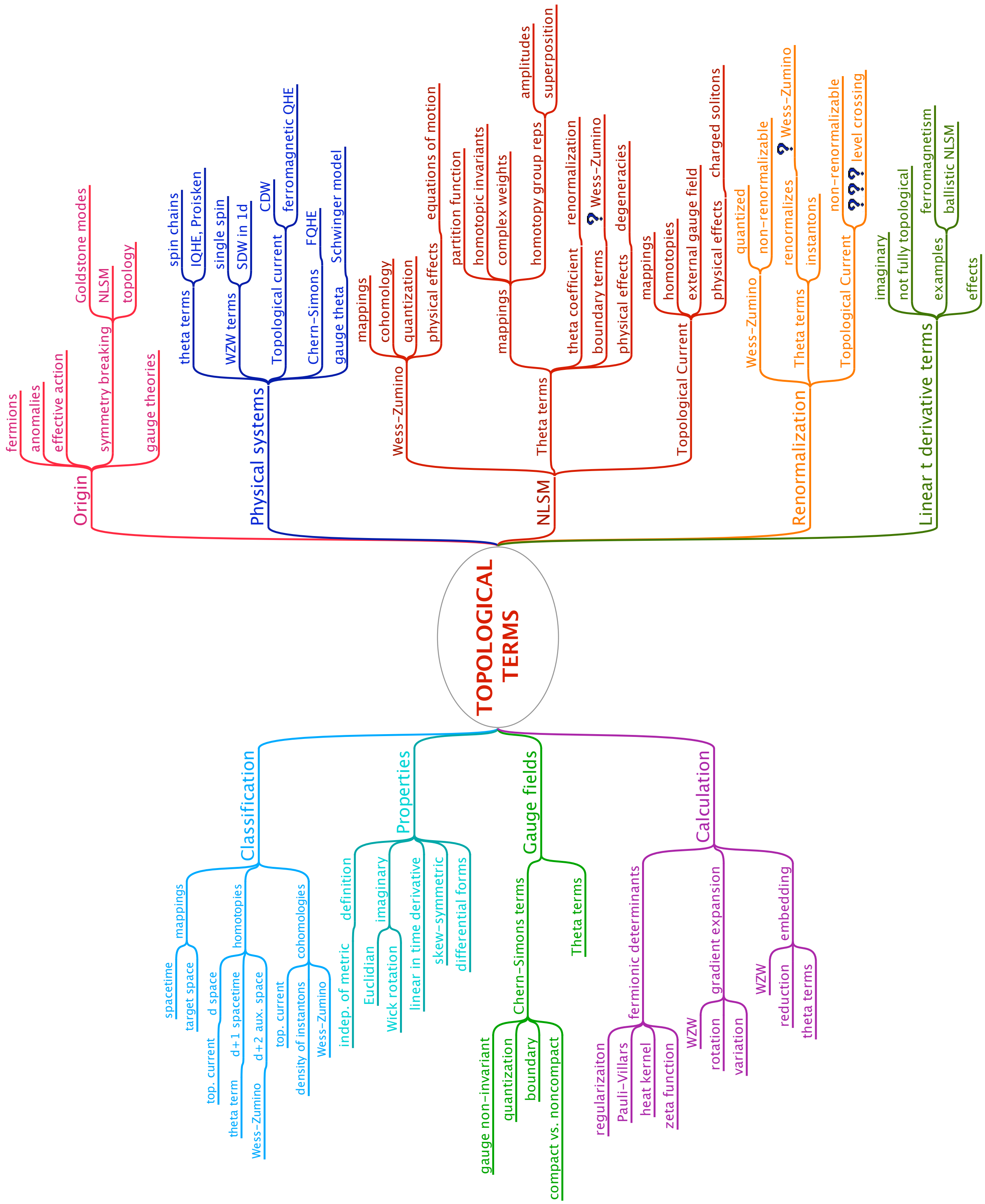
Second remark is that there are other mechanisms in addition to the spontaneous symmetry breaking which result in low energy excitations. One of the most important ones is realized when local (or gauge) symmetry is present†. Then, gauge invariance plus locality demands the presence of massless particles (e.g, photons) in the system. Therefore, the low energy theories are often gauge theories. Similar, to an explicit symmetry breaking in case of Goldstone particles there are some mechanisms which generate masses for gauge bosons. These are, e.g., Higgs mechanism and confinement of gauge fields. We will have some examples of topological terms made out of gauge fields in this review although our main focus will be on non-linear sigma models‡.

1.2 Course structure

the non-linear realization of symmetries in this model. E.g., constraint $\vec{n}^2 = 1$ is non-linear. Sigma (σ) is a historic notation for the "order parameter" in theories of weak interactions.

† Sometimes there are massless fermionic modes at low energis "protected" by some mechanism. The theory of such systems is under construction[**?**] and in this review we concentrate exclusively on bosonic effective theories.

‡ Non-linear sigma models and gauge theories has a lot in common[3]


TOPOLOGICAL TERMS
Origin
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effective action
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spin chains
IQHE, Proisken
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NLSM
Wess–Zumino
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