4
Homotopy theory primer

Given that some topological invariant is different for topological spaces X and Y one can definitely say that the spaces are not homeomorphic. The more invariants one has at his/her disposal the more detailed testing of equivalence of X and Y one can perform. The homotopy theory constructs infinitely many topological invariants to characterize a given topological space. The main idea is the following. Instead of directly comparing structures of X and Y one takes a "test manifold" M and considers the spacings of its mappings into X and Y, i.e., spaces $C(M, X)$ and $C(M, Y)$. Studying *homotopy classes* of those mappings (see below) one can effectively compare the spaces of mappings and consequently topological spaces X and Y.

It is very convenient to take as "test manifold" M spheres S^n. It turns out that in this case one can endow the spaces of mappings (more precisely of homotopy classes of those mappings) with group structure. The obtained groups are called *homotopy groups* of corresponding topological spaces and present us with very useful topological invariants characterizing those spaces.

In physics homotopy groups are mostly used not to classify topological spaces but spaces of mappings themselves (i.e., spaces of field configurations).

4.1 Homotopy

<u>Definition</u> Let $I = [0, 1]$ is a unit closed interval of R and $f : X \to Y$, $g : X \to Y$ are two continuous maps of topological space X to topological space Y. We say that these maps are *homotopic* and denote $f \sim g$ if there exists a continuous map $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. The map F itself is called a *homotopy* between f and g.

$\sim$ is an equivalence relation in the space of mappings (symmetric, reflective, and transitive) and therefore all mappings can be divided in *homotopy classes* of homotopic mappings.

4.2 Zeroth homotopy group

Let us start with mappings of a single point into a topological space X. A set of homotopy classes of these mappings is denoted $\pi_0(X)$ and is sometimes called *zeroth homotopy group*†.

It is easy to see that if the mappings $f(\cdot)$ and $g(\cdot)$ belong to the same connected component of X the mappings $f \sim g$ are homotopic. Therefore, $\pi_0(X)$ is nothing else but the set of connected components of X. Zeroth homotopy group is trivial $\pi_0(X) = 0$ iff X is a connected space. For example,

$$\pi_0(SO(3)) = 0, \tag{4.1}$$
$$\pi_0(O(3)) = Z_2. \tag{4.2}$$

Here Z_2 is a group consisting of 1 and -1 with multiplication as a group operation

$$Z_2 = \{1, -1\}.$$

4.3 Fundamental group

A *fundamental group* or a *first homotopy group* is a group of homotopy classes of closed paths with a given *base point*.

4.3.1 Paths

- Directed path
- Product of paths
- Inverse path
- Equivalence of paths
- Set of classes of equivalent paths
- Group structure on the set of homotopy classes of equivalent paths
- Fundamental group
- Homotopic invariance of the fundamental group
- Independence on the base point
- Examples

† The name is misleading as generally there is no group structure in $\pi_0(X)$. However, in some cases, e.g., when X is group itself, $\pi_0(X)$ can be a group.

4.3.2 Homotopy type

Definition: Topological spaces X and Y are of the same *homotopy type* ($X \simeq Y$), if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \sim id_Y$ and $g \circ f \sim id_X$.

$\simeq$ is an equivalence relation in the set of topological spaces

If X and Y are homeomorphic, they are also of the same homotopy type but the converse is not true. For example the line is of the same homotopy type as a single point $R \simeq \cdot$ but they are not homeomorphic.

Two topological spaces of the same homotopy type have the same fundamental group.

Theorem: Let X and Y be topological spaces of the same homotopy type. If $f : X \to Y$ is a homotopy equivalence, $\pi_1(X, x_0)$ is isomorphic to $\pi_1(Y, f(x_0))$.

In fact, generally if $X \simeq Y$ there is a one-to-one correspondence between homotopy classes of continuous mappings from a topological space K to X (i.e., $[K, X]$) and $[K, Y]$.

It immediately follows from this theorem that the fundamental group is invariant under homeomorphisms, and hence is a topological invariant.

A very useful concept for calculation of homotopy groups is a *deformation retract*. A subspace $R \subset X$ is a *deformation retract* of X if there exists a homotopy of $id_X : X \to X$ and $f : X \to R$ such that all points of R are fixed during the deformation.

R and X are of the same homotopy type $R \simeq X$ and, therefore, their fundamental groups are isomorphic.

For example a circle S^1 is a deformation retract of the plane with a punched point $R^2 \backslash \cdot$ and therefore

$$\pi_1(R^2 \backslash \cdot) = \pi_1(S^1) = Z.$$

If a point of X is a deformation retract of X the space X is called *contractible* (space X can be contracted to a point). In this case the fundamental group of X is trivial $\pi_1(X) = 0$. If connected space X has a trivial fundamental group $\pi_1(X)$ it is called *simply connected*.

4.4 Fundamental group: examples and simple properties

Let us list some examples of fundamental groups of different manifolds.

4.4.1 R^n, C^n, and D^n

All these spaces are contractible and, therefore, have trivial fundamental group

$$\pi_1(R^n) = \pi_1(C^n) = \pi_1(D^n) = 0. \tag{4.3}$$

4.4.2 Spheres S^n for $n > 1$

Every loop on S^n with $n > 1$ can be contracted to a point. Therefore,

$$\pi_1(S^n) = 0, \quad \text{for } n > 1. \tag{4.4}$$

4.4.3 Circle S^1

As we already know from the example of a particle on a ring the fundamental group of circle is the group of integer numbers

$$\pi_1(S^1) = Z. \tag{4.5}$$

The mappings $\phi : S^1 \to S^1$ can be divided in topological classes so that two mappings can be smoothly deformed one into another if and only if they belong to the same topological class. These mappings are labeled by integer number W – the winding number of the mapping and form a group

$$W = \int_0^\beta \frac{dt}{2\pi}\, \partial_t \phi, \tag{4.6}$$

where $\phi(t)$ is defined on $t \in [0, \beta]$ with periodic boundary condition $e^{i\phi(\beta)} = e^{i\phi(0)}$.

The product $\gamma_2 \cdot \gamma_1$ of paths γ_1 and γ_2 characterized by W_1 and W_2 is characterized by $W = W_1 + W_2$.

4.4.4 Rubber band around the pole $R^2 \backslash R^1$

The space $R^2 \backslash R^1$ is of the same homotopy type as S^1 (S^1 is a deformation retract of that space). Therefore,

$$\pi_1(R^2 \backslash R^1) = \pi_1(R^2 \backslash \cdot) = \pi_1(S^1) = Z. \tag{4.7}$$

4.4.5 Plane with two punctures and bouquet of circles $R^2 \backslash \{x_1, x_2\} \simeq S^1 \vee S^1$

The two-dimensional plane R^2 with two punctures (subtracted points x_1 and x_2 has a bouquet of two circles as a deformation retract. The bouquet of circles is the space made out of several circles with one common point. It is easy to see that

$$\pi_1(R^2 \backslash \{x_1, x_2\}) = \pi_1(S^1 \vee S^1) = \mathrm{Free}(a, b). \tag{4.8}$$

Here Free(a, b) is a *free group* generated by two generators a and b. Each *word*: $a^{n_1} b^{m_1} \dots a^{n_k} b^{m_k}$ is a new element of the group with product defined as a concatenation of words. One can take as generators, e.g., the classes of the path with winding number 1 going around the first circle and a similar path around the second circle.

Notice that this fundamental group is non Abelian: ab and ba are different words.

4.4.6 Free homotopy classes of loops and importance of the base point

The presence of the base point x_0 in the definition of the fundamental group is essential although the fundamental group itself does not depend on that base point (for connected X).

If one omits the base point one should consider instead of a fundamental group the set of homotopy classes of mappings of S^1 into X, i.e., the set $[S^1, X]$. This set in general does not have a group structure. It is called *free homotopy classes of loops* on space X.

4.4.7 Real projective plane RP^2

$$\pi_1(RP^2) = \pi_1(S^2/Z_2) = Z_2. \tag{4.9}$$

4.4.8 The free action of a discrete group on a simply connected space

One can generalize the example of RP^2 to the case where some discrete group Γ freely acts on a simply connected topological space X. In this case

$$\pi_1(X/\Gamma) = \Gamma, \qquad \text{if } \pi_1(X) = 0. \tag{4.10}$$

4.4.9 RP^n

$$\pi_1(RP^n) = \pi_1(S^n/Z_2) = Z_2. \tag{4.11}$$

4.4.10 Two- and higher-dimensional tori

The mappings (topological classes) of circle into two-dimensional torus are labeled by two integers – two winding numbers of circle around torus cycles (see Fig.). One can write

$$\pi_1(T^1) = \pi_1(S^1 \times S^1) = Z \oplus Z. \tag{4.12}$$

In general if manifold M is a direct product $M = M_1 \times M_2$ then it is true that

$$\pi_1(M) = \pi_1(M_1 \times M_2) = \pi_1(M_1) \oplus \pi_1(M_2).$$

Applying this rule recursively to the n-dimensional torus $T^{\mathrm{n}} = \overbrace{S^1 \times S^1 \times \ldots \times S^1}^{\mathrm{n}}$. we obtain

$$\pi_1(T^{\mathrm{n}}) = \overbrace{Z \oplus Z \oplus \ldots \oplus Z}^{\mathrm{n}}. \tag{4.13}$$

4.4.11 Compact groups

One can think of $SO(3)$ as of three-dimensional ball of radius π so that each point of this ball corresponds to a rotation around radius-vector of that point over the angle equal to the length of that radius-vector. Points of this ball are in almost one to one correspondence with the rotations of three-dimensional space except for the points on the surface of the ball. The points on surface correspond to the angle π rotations around different axes and, obviously, π rotations around oppositely directed axes (i.e., rotations by π and $-\pi$ around the same axis) are the same. Therefore, the opposite points on the surface of the ball must be identified. The manifold obtained in this identification is called RP^3 (three dimensional real projective space). There are only two topological classes of mapping of circle into such a manifold. They correspond to the mappings shown in Fig. We write:

$$\pi_1(SO(3)) = \pi_1(RP^3) = Z_2. \tag{4.14}$$

Here Z_2 is a group consistent of two elements. It can be thought of the group of integers modulo 2 with respect to addition or as of the group $\{1, -1\}$ with respect to multiplication.

4.4.12 Non-Abelian fundamental group. Riemann surface of genus 2

So far all considered fundamental groups were Abelian groups. This is not always the case and here we give an example of non-Abelian fundamental group. If $M_{g=2}$ is a two-dimensional Riemann surface of genus 2 (sphere with two handles) we have

$$\pi_1(M_{g=2}) = \{a_i, b_i; i = 1, \dots 4\}/\{a_1 b_1 a_2 b_2 = b_4 a_4 b_3 a_3\}. \tag{4.15}$$

Here $\{a_i, b_i; i = 1, \dots 4\}$ is a free group generated by eight generators and identification $a_1 b_1 a_2 b_2 = b_4 a_4 b_3 a_3$ is performed in (4.15). We refer the reader to textbooks in bibliography to find out how this result was obtained as well as for generalization to the Riemann surfaces of higher genus.

4.4.13 Non-Abelian fundamental group. Klein bottle.

$\pi_1(K) = \{x, y\}/\{xyxy^{-1} = 1\}$.

4.5 Fractional statistics of quantum particles

In this section I follow Ref. [Wu], where more details can be found.

The configuration space of n identical particles in d-dimensional space R^d is given by

$$K = (R^{nd} \backslash D)/P_n. \tag{4.16}$$

Here, P_n is a permutation group of n elements and D is the "diagonal" – configurations where positions of at least two particles coincide. D is defined as $\{(r_1, \dots, r_n) | r_i = r_j \text{ for some } i \neq j\}$. We have subtracted the diagonal D because we assume that particles are not penetrable.

Topological theta term exists if the fundamental group of the configuration space is non-trivial. We have $\pi_1(R^{nd} \backslash D) = 0$ for $d \geq 3$. Therefore,

$$\pi_1(K) = P_n, \qquad \text{for } d \geq 3. \tag{4.17}$$

The only one-dimensional unitary irreducible representations of permutation group P_n are totally symmetric representation (all elements are represented by unity) and totally antisymmetric (element $\sigma \to (-1)^{P(\sigma)}$, where $P(\sigma)$ is the parity of the permutation σ). The former corresponds to Bose statistics of particles while the latter to Fermi statistics. We conclude that in spatial dimension three and higher the only allowed statistics are Bose and Fermi.

However, in two dimensions $d = 2$ the space $R^{nd} \backslash D$ is not simply connected. In this case

$$\pi_1(K) = B_n, \qquad \text{for } d = 2, \tag{4.18}$$

where B_n is the *braid group.*

The braid group of n strands can be formally defined as the group formed by $n-1$ generators σ_j, $j = 1, 2, \ldots, n-1$ with the following relations (braid or Artin relations) (i) $\sigma_i \sigma_j = \sigma_j \sigma_i$, if $|i-j| \geq 2$, (ii) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for $i = 1, 2, \ldots, n-2$.

It immediately follows from these relations that the one-dimensional unitary irreducible representations of B_n are labeled by $e^{i\theta}$ with $\theta \in [0, 2\pi)$ so that $\sigma_i \to e^{i\theta}$ for $i = 1, 2, \ldots, n-1$.

Particular cases $e^{i\theta} = \pm 1$ $(\theta = 0, \pi)$ correspond to previously considered Bose and Fermi statistics of particles respectively. In general, the statistics of particles in two spacial dimensions can interpolate between Bose and Fermi statistics and is called θ-statistics (or anyonic statistics).

4.6 Higher homotopy groups

Let us consider a disk D^n with the boundary $\partial D^n = S^{n-1}$. Topologically one can think of the disk as of hypercube $I \times I \ldots \times I = I^n$, where $I = [0, 1]$. We consider the continuous mappings of the disk D^n into the topological space X, i.e., $f : D^n \to X$ subject to a condition that the boundary of the disk is mapped onto a fixed point $f(\partial D^n) = x_0 \in X$.

A set of homotopy classes of such mappings forms *n-th homotopy group* of the topological space X and is denoted $\pi_n(X; x_0)$.

If $(t_1, \ldots, t_n) \in I^n$ we define a group structure on $\pi_n(X; x_0)$ through the representative mappings of homotopy classes defining the product

$$f_2 \cdot f_1(t) = \begin{cases} f_1(2t_1, t_2, \ldots, t_n), & \text{if } t_1 \leq 1/2, \\ f_2(2t_1 - 1, t_2, \ldots, t_n), & \text{if } t_1 \geq 1/2, \end{cases} \tag{4.19}$$

the unity

$$e(t) = x_0, \tag{4.20}$$

and the inverse element

$$f^{-1}(t) = f(1 - t_1, t_2, \ldots, t_n). \tag{4.21}$$

One can prove that the above definitions of group operations is preserved by homotopies and, therefore, define a group structure on $\pi_n(X; x_0)$.

Theorem: Higher homotopy groups ($\pi_n(X; x_0)$ for $n \geq 2$) are **Abelian groups**.

One can prove this theorem showing that for any mappings α and β from D^n to X with boundary mapped to x_0 there is a homotopy of the mapping $\alpha \cdot \beta$ to the mapping $\beta \cdot \alpha$.

As in the case of fundamental group one can show that the higher homotopy groups do not depend on the choice of the base point x_0 provided the space X is connected. Therefore, one can use the notation $\pi_n(X)$.

Also we have the following theorem.

Theorem: If topological spaces X and Y are of the same homotopy type $X \simeq Y$ all their homotopy groups are the same (isomorphic) $\pi_n(X) = \pi_n(Y)$ for all $n = 1, 2, \ldots$.

For the direct product of topological spaces we have

$$\pi_n(X \times Y) = \pi_n(X) \times \pi_n(Y). \tag{4.22}$$

4.7 Higher homotopy groups: examples and simple properties

4.7.1 R^n*,* C^n*, and* D^n

All these spaces are contractible and, therefore, have trivial homotopy groups

$$\pi_i(R^n) = \pi_i(C^n) = \pi_i(D^n) = 0. \tag{4.23}$$

This is true for $i \geq 1$.

4.7.2 Topology of S^2

Consider the mapping $S^2 \to S^2$ explicitly given by $\vec{n}(t, \rho)$ with $\vec{n} \in S^2$ parameterizing the target sphere and t, ρ are the local coordinates on the first sphere. There is an infinite number of homotopy classes of such mappings labeled by a topological invariant *degree of mapping* - an integer number k given by:

$$k = \int_{S^2} d^2x \frac{1}{8\pi} \epsilon^{\mu\nu} \vec{n} \cdot [\partial_\mu \vec{n} \times \partial_\nu \vec{n}] \, . \tag{4.24}$$

One can easily show that the variation of k with respect to $\vec{n}(t, \rho)$ is zero. This means that the value k does not change under small variations of $\vec{n}$ i.e., it is a topological invariant.

Let us consider the integrand around the point in x-space where $\vec{n} = (0, 0, 1)$. We write $\vec{n} = (u_1, u_2, 1)$ and obtain $q(\vec{x}) = \frac{1}{8\pi} \epsilon^{\mu\nu} \vec{n} \cdot [\partial_\mu \vec{n} \times \partial_\nu \vec{n}] = \frac{1}{4\pi} \epsilon^{\mu\nu} \partial_\mu u_1 \partial_\nu u_2 = \frac{1}{4\pi} \frac{\partial(u_1, u_2)}{\partial(t, \rho)}$. One easily recognizes in the last expression the Jacobian of a transformation from the local coordinates ρ, t to the local coordinates u_1, u_2 on the target sphere. Therefore k is the number of times the vector $\vec{n}$ sweeps over the target sphere S^2. This shows that k is an integer.

It can be proven that two mappings of the same degree k are homotopic.

Also, the degrees of mappings are added if mappings are multiplied. Therefore,

$$\pi_2(S^2) = Z. \tag{4.25}$$

4.7.3 Homotopy groups of spheres $\pi_i(S^n)$ for $i < n$

It is easy to argue that every i-loop on S^n with $i < n$ can be contracted to a point. Therefore,

$$\pi_i(S^n) = 0, \quad \text{for } i < n. \tag{4.26}$$

4.7.4 Homotopy groups of spheres $\pi_i(S^n)$

The formula (4.25) can be generalized to higher dimensions. The homotopy classes of $S^n \to S^n$ are labeled by the degree of mappings and form a group of integer numbers

$$\pi_n(S^n) = Z. \tag{4.27}$$

The explicit formula (4.24) can also be generalized

$$k = \frac{1}{A_n} \int_{S^n} d^n x \, \epsilon^{\mu_1 \ldots \mu_n} \pi^0 \partial_{\mu_1} \pi^1 \ldots \partial_{\mu_n} \pi^n. \tag{4.28}$$

Here $(\pi^0, \pi^1, \ldots, \pi^n) \in S^n$ $(\vec{\pi}^2 = 1)$ and A_n is some normalization constant (fix it!).

The calculation of homotopy groups of spheres $\pi_i(S^n)$ for $i > n$ is a very difficult problem which stimulated the development of topology but have not been solved in general to this point. One of highly non-trivial results is

$$\pi_3(S^2) = Z. \tag{4.29}$$

This result belongs to Hopf and the integer number labelling homotopy classes in this case is called Hopf invariant (see Sec.**??** for more details). The integer number is a *linking number* of preimages of two arbitrary points from the target S^2.

In these lectures we use only some homotopy groups of spheres. We list these groups for convenience in the following table.

	S^1	S^2	S^3	S^4	S^5	S^6	S^7
π_1	Z	0	0	0	0	0	0
π_2	0	Z	0	0	0	0	0
π_3	0	Z	Z	0	0	0	0
π_4	0	Z_2	Z_2	Z	0	0	0
π_5	0	Z_2	Z_2	Z_2	Z	0	0
π_6	0	Z_{12}	Z_{12}	Z_2	Z_2	Z	0
π_7	0	Z_2	Z_2	$Z \oplus Z_{12}$	Z_2	Z_2	Z

The important property, so-called *stability* of homotopy groups, is that $\pi_{n+k}(S^n)$ is the same for all $n \geq k+2$.

4.7.5 Homotopy groups of Lie groups

Some examples of homotopy groups of Lie groups are given in the appendix 1.

4.7.6 Other spaces

I have listed examples of homotopy groups of some topological spaces used in physics in the appendix 1.

4.8 Exercises

Exercise 1: Topological invariant: $S^2 \to S^2$

Consider a three-dimensional unit vector field $\vec{n} \in S^2$ on a two-dimensional plane $\vec{n}(x,y)$ with constant boundary conditions $\vec{n}(x,y) \to \hat{e}_3$ as $(x,y) \to \infty$. Show that

$$Q = \int d^2x \, \frac{1}{8\pi} \epsilon^{\mu\nu} \vec{n}[\partial_\mu \vec{n} \times \partial_\nu \vec{n}] \tag{4.30}$$

is an integer-valued topological invariant. Namely,

a) Show that under small variation $\delta\vec{n}$ of a vector field the corresponding variation $\delta Q = 0$.

b) Show that the integrand in (4.31) is a Jacobian of the change of variables from x, y to a sphere $\vec{n}$ and it is normalized in such a way that the area of the sphere is

1. Therefore, Q is an integer degree of mapping of a plane (with constant boundary conditions) onto a sphere.

Hint: In b) consider the vicinity of the northern pole of the sphere only and extend your result to the whole sphere by symmetry.

Exercise 4.2: Topological invariant: $S^3 \to S^3$ Consider a three-dimensional unit vector field $\vec{\pi} \in S^3$ on a three-dimensional space $\vec{\pi}(x, y, z)$ with constant boundary conditions $\vec{\pi}(x, y, z) \to (0, 0, 0, 1)$ as $(x, y, z) \to \infty$. Show that

$$Q = A \int d^2x \, \epsilon^{\mu\nu\lambda} \epsilon^{abcd} \pi^a \partial_\mu \pi^b \partial_\nu \pi^c \partial_\lambda \pi^d \tag{4.31}$$

is an integer-valued topological invariant with properly chosen normalization constant A. Namely,

a) Show that under small variation $\delta\vec{\pi}$ of a vector field the corresponding variation $\delta Q = 0$.

b) Show that the integrand in (4.31) is a Jacobian of the change of variables from x, y, z to a sphere $\vec{\pi}$ up to normalization.

c) Choose A so that it is normalized in such a way that the area of the 3-sphere is 1. Therefore, Q is an integer degree of mapping of a space (with constant boundary conditions) onto a 3-sphere.

Hint: In b) consider the vicinity of the northern pole of the 3-sphere only and then extend your result to the whole 3-sphere by symmetry.

Exercise 4.3: Topological invariant: $S^n \to S^n$ Generalize the results of previous exercises. Write down an explicit formula for the degree of mapping $S^n \to S^n$ with correct normalization factor.

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