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Topological defects and textures in ordered media

In this chapter we consider how to classify topological defects and textures in ordered media. We give here only a very short account of the method following Ref. [6], where the reader can found all necessary details.

5.1 Spontaneous symmetry breaking and order parameter space

Suppose that there is a quantum field theory (QFT) with symmetry group G. It means here that the action (Hamiltonian, Lagrangian, etc.) is invariant under symmetry transformations from G. If there is a field which acquires an expectation value which is not invariant under the symmetry G we say that the symmetry G is *spontaneously broken*. The field is called the *order parameter field*. If the expectation value of the order parameter field (which is referred to as *order parameter*) has a residual symmetry H (subgroup of G) we call $M = G/H$ the *order parameter space* (or manifold).

Every point of the order parameter space M represents a vacuum (or thermodynamic state). The energies (or free energies) of the corresponding vacua (or thermodynamic states) are the same because G is the exact symmetry of QFT Hamiltonian.

In local QFT it is natural to expect that in the state with broken symmetry the states corresponding to slowly changing (in space) order parameter are the low energy states. Therefore, the small gradient configurations $m(x) : R^d \to M$ parameterize low energy states of the field theory in d spatial dimensions. One describes the low energy sector of QFT (or statistical mechanics) in the phase with broken symmetry by *nonlinear sigma models* with actions (energies) depending on gradients of the order parameter $\int d^dx\,(\nabla m)^2$. We assume that the mappings $m(x)$ are smooth almost everywhere. The mappings $R^d \to M$ are referred to as *nonuniform states*, while the mappings which map the whole R^d to a single point $m_0 \in M$ are *uniform states* with the value of the order parameter m_0.

5.2 Topological defects and homotopy theory

It is useful to consider not only mappings $m(x)$ which are smooth everywhere but also the ones with singularities. If $m(x)$ is smooth for all $x \in R^d$ except for isolated points (lines, surfaces), we say that $m(x)$ is a configuration with point (line, surface) defect. The configurations with defects typically have higher energies than everywhere smooth configurations. Notice also, that in the vicinity of the singularity the validity of nonlinear sigma model description is questionable and typically the degrees of freedom other than fluctuations of the order parameter become relevant near defects.

The relevance of a particular type of defect in the low energy description of the theory is determined dynamically (e.g. studying the effect of topological defects on a particular correlation function in renormalization group approach). There is however, a class of defects which if present can not be removed by continuous deformation of fields because of the non-trivial topology of the order parameter space. Such defects are called *topological defects*.

As an example, consider a vortex in a superfluid film. The order parameter in this case is a superfluid phase $\phi \in S^1$ and the vortex configuration is given, e.g., by $R^2 \to S^1$ with $\phi(x) = \arg(\vec{x})$. Let us enclose the defect by a circle of the large radius and notice that the configuration $\phi(x)$ induces a continuous mapping $S^1 \to S^1$ given by $\phi(\theta) : |\vec{x}|e^{i\theta} \to \theta$. This mapping has the winding number one. The non-vanishing winding number means that there is a non-removable (topological) defect inside the unit circle we introduced. This defect can not be removed by local deformations of $\phi(x)$ inside the circle. One can only move the defect outside of the circle. The possibility of having topological point defects (vortices in a superfluid film) is, therefore, following from the non-triviality of the fundamental group $\pi_1(S^1) = Z$. Here π_1 is taken because the surface surrounding the defect is one-dimensional sphere (circle) and the argument of $\pi_1(S^1)$ is the order parameter space for a superfluid - S^1. Moreover, the structure of the homotopy group (Z in this case) tells us how composite defects can be constructed. For example, merging two topological defects characterized by winding numbers W_1 and W_2 one obtains the topological defects with winding number $W_1 + W_2$.

The above construction can be straightforwardly extended to a more general case. We consider the order parameter space M in d-dimensional space. One needs $(d-2)$-dimensional sphere S^{d-2} to enclose the line defect in d dimensions. Therefore, the possibility to have topological line defects requires that $\pi_{d-2}(M) \neq 0$. We summarize this type of topological analysis as

$$\begin{aligned} \pi_{d-1}(M) &\neq 0, \text{ point defects}, \\ \pi_{d-2}(M) &\neq 0, \text{ line defects}, \\ \pi_{d-3}(M) &\neq 0, \text{ surface defects}, \\ &\dots \end{aligned} \tag{5.1}$$

As one more example, let us consider the possibility of having domain walls (surface defects) in three dimensions. It is quite obvious that the surface defect is not removable only if the values of the order parameter on both sides of the defect belong to different connected components of the order parameter space. Therefore, we should check for non-triviality of zeroth homotopy group $\pi_0(M) \neq 0$ which agrees with (5.1).

5.3 Topological textures

Let us now consider smooth order parameter configurations (no defects). As an example we take the $O(3)$ nonlinear sigma model in 2d. The energy is given by

$$E = \frac{1}{2g} \int d^2x \, (\partial_\mu \vec{n})^2, \tag{5.2}$$

where $\vec{n} \in S^2$ ($\vec{n}^2 = 1$). The energy is finite if asymptotically

$$\vec{n}(\vec{x}) \longrightarrow \vec{n}_0, \text{ as } |\vec{x}| \to \infty, \tag{5.3}$$

where $\vec{n}_0$ is a constant unit vector which does not depend on the direction in which the limit is taken. One can see that the neighborhood of infinity in R^2 is mapped into the neighborhood of n_0.

The configurations $\vec{n}(\vec{x})$ with constant boundary condition (5.3) are specified by the mapping $S^2 \to M$, where S^2 here is compactified space R^2 and $M = S^2$ is the order parameter space. These mappings may be non-trivial since $\pi_2(S^2) = Z \neq 0$. One can have non-trivial smooth configurations of $\vec{n}(\vec{x})$ which can not be removed without breaking constant boundary conditions, i.e., going through the infinite energy barrier. Such configurations are called *textures*.

The topological test for textures in spacial dimension d, therefore, is the non-triviality of the homotopy group

$$\pi_d(M) \neq 0, \text{ textures}. \tag{5.4}$$

For space dimensions that occur most often in condensed matter applications we have the following table

	$d=1$	$d=2$	$d=3$	$d=4$
π_0	point	line	surface	hypersurface
π_1	texture	point	line	surface
π_2		texture	point	line
π_3			texture	point
π_4				texture

5.4 Phase transitions driven by topology: BKT transition

In spite of their high energies topological defects might give a very important contribution to the partition function and even change the phase of matter. The classic example of such topological defect driven phase transition is the Berezinskii-Kosterlitz-Thouless (BKT) transition in two-dimensional XY model. We refer the reader to Refs. [3, 4] for details on XY model.

The energy of the vortex defect in this model diverges logarithmically with the size of the system $E \sim J\log(L)$. and at low temperature vortices exist only in the bound form with anti-vortices (molecular phase). At higher temperature, though, the entropy of the vortex is also proportional to the logarithm $\log(L)$ and at the temperature higher than critical $T_c \sim J$ the free energy of the vortex becomes negative $F \sim J\log(L) - T\log(L) < 0$ and vortices proliferate (plasma or Debye phase).

As a result of the proliferation of point defects the behavior of correlation functions changes from power law decay at low temperature phase to the exponential decay at high temperatures.

5.5 Exercises

Exercise 1: Nematic

Nematic is a liquid crystal characterized by an order parameter which is the unit three-component vector $\vec{n} = (n_1, n_2, n_3)$, $\vec{n}^2 = 1$ with an additional condition $\vec{n} \sim -\vec{n}$. The latter means that two unit vectors which are opposite to each other describe the same state.

What types of topological defects and textures are allowed for three-dimensional nematic? What about two-dimensional one?

Exercise 2: Crystal

One can view a crystalline state as continuous translational symmetry broken to the subgroup of discrete translations. Then the order parameter space should be identified (for three-dimensional crystal) with $M = G/H = R^3/(Z \times Z \times Z)$.

a) What (geometrically) is the order parameter space for this system?

b) What are the homotopy groups of this manifold $\pi_{0,1,2,3}(M)$?

c) What types of topological defects and textures are allowed in such a system?

Exercise 3: Superfluid $^3He - A$

The order parameter of superfluid $^3He - A$ can be represented by two mutually orthogonal unit vectors $\vec{\Delta}_1$, $\vec{\Delta}_2$. That is, at each point in three-dimensional space one has a pair of vectors with properties $\vec{\Delta}_1^2 = \vec{\Delta}_2^2 = 1$ and $\vec{\Delta}_1 \cdot \vec{\Delta}_2 = 0$.

a) What is the manifold of degenerate states for this system?

b) What are the homotopy groups of this manifold $\pi_{0,1,2,3}(M)$?

c) What types of topological defects and textures are allowed in such a system?

Exercise 4: Heisenberg model

What topological defects and textures one should expect in the ordered state of a three-dimensional classical Heisenberg model? What changes if the order parameter is a director instead of a vector? A “director” means a vector without an arrow, i.e., one should identify $\vec{S} \equiv -\vec{S}$. The models with a director as an order parameter are used to describe nematic liquid crystals.

Exercise 5: Continuum limit of XY *model*

Let us start with the XY model defined on a cubic d-dimensional lattice. The allowed configurations are parameterized by a planar unit vector $\vec{n}_i = (\cos\theta_i, \sin\theta_i)$ on each site i of the latttice. The energy is given by

$$E = -\sum_{\langle ij \rangle} J \cos(\theta_i - \theta_j). \tag{5.5}$$

We assume that the most important configurations are smooth on a lattice scale and one can think of θ_i as of smooth function $\theta(\vec{x})$ defined in R^d - continuous d-dimensional space. Show that the energy is given in this continuous limit by

$$E = \frac{J}{2} \int \frac{d^d x}{a^d} \, a^2 (\partial_\mu \theta)^2, \tag{5.6}$$

where a is the lattice constant. The combination $\rho_s^{(0)} = Ja^{2-d}$ is referred to as *bare spin-wave stiffness* (or *bare superfluid density*).

Compute the energy of the vortex in such a model. Remember that the divergent integrals should be cut by lattice constant a and by the size of the system L at small and large distances respectively.

Exercise 6: Correlation function $\langle(\theta(x) - \theta(0))^2\rangle$

Calculate the correlation function $\langle(\theta(x)-\theta(0))^2\rangle$ in the XY model in d dimensions neglecting the topology of θ, i.e., neglect vortices and think about θ as of real number without periodicity. Divergencies at small distances should be cut off by the lattice constant a.

Exercise 7: Correlation function $\langle\vec{n}(x)\vec{n}(0)\rangle$

Using the result of the previous exercise calculate the correlation function $\langle\vec{n}(x)\vec{n}(0)\rangle$ in the XY model in d dimensions neglecting the topology of θ. Write $\langle\vec{n}(x)\vec{n}(0)\rangle = \langle\cos(\theta(x) - \theta(0))\rangle = \mathrm{Re}\,\langle e^{i(\theta(x)-\theta(0))}\rangle$ and use the properties of Gaussian integrals.

Make the conclusion about the existence of a true long range order in XY model in 2d and relate it to Mermin-Wagner theorem.

Exercise 8: Correlation function $\langle\vec{n}(x)\vec{n}(0)\rangle$

Let us now start with high temperatures. Assume that $J/T \ll 1$. Using high temperature expansion show that correlation function $\langle\vec{n}(x)\vec{n}(0)\rangle$ decays exponentially. Find the correlation length at high temperatures.

Exercise 9: Vortex unbinding

Make an estimate of the BKT phase transition temperature in 2d XY model. Use the energy of the vortex calculated previously, the estimate of the entropy of the vortex, and the condition $F = 0$ for the free energy of the vortex.

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