7

WZW term in quantum mechanics: single spin

Wess-Zumino introduced an effective Lagrangian to summarize the anomalies in current algebras[5]. E. Witten considered global (topological) aspects of this effective action[12]. Simultaneously, S. P. Novikov studied multi-valued functionals[14].

7.1 Quantum spin

Let us consider a simple example of how Wess-Zumino effective Lagrangian appears from the "current algebra". To simplify the story we take an example of quantum spin S. This is a quantum mechanical system with an $SU(2)$ spin algebra playing the role of "current algebra" of quantum field theory. We have standard spin commutation relations

$$[S^a, S^b] = i\epsilon^{abc}S^c, \tag{7.1}$$

where a, b, c take values x, y, z. We require that

$$\vec{S}^2 = S(S+1), \tag{7.2}$$

where $2S$ is an integer number defining the representation (the value of spin). Let us consider the simplest possible Hamiltonian of a quantum spin in a constant magnetic field

$$H = -\vec{h} \cdot \vec{S} \tag{7.3}$$

and derive an operator equation of motion

$$\partial_t \vec{S} = i[H, \vec{S}] = -i[\vec{h} \cdot \vec{S}, \vec{S}] = \vec{S} \times \vec{h}. \tag{7.4}$$

In the classical limit $S \to \infty$ (or $\hbar \to 0$) it is convenient to write $\vec{S} \to S\vec{n}$ so that $\vec{n}$ is a classical unit vector $\vec{n}^2 = 1$ and equation of motion(7.4) becomes classical equation of motion

$$\partial_t \vec{n} = \vec{n} \times \vec{h}. \tag{7.5}$$

The natural question immediately occurs is what classical action corresponds to these equation of motion. It turns out that writing down this action is not completely trivial problem if one desires to keep an explicitly $SU(2)$ invariant form. Let us first derive it using non-invariant parameterization in terms of spherical angles $\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. We assume that the angle θ is measured from the direction of magnetic field $\vec{h} = (0,0,h)$. The Hamiltonian (7.3) becomes $H = -hS\cos\theta$ and equations of motion (7.5) become $\dot{\phi} = -h$ and $\dot{\theta} = 0$ – precession around the direction of magnetic field. We obtain this equations as Hamiltonian equations identifying the momentum conjugated to ϕ coordinate as

$$p_\phi = -S(1-\cos\theta). \tag{7.6}$$

Then the classical action of a single spin in magnetic field can be written as One can check that the answer is given by

$$S[\vec{n}] = -4\pi S W_0 + \int dt\, S\vec{h}\cdot\vec{n}, \tag{7.7}$$

where W_0 is defined using a particular choice of coordinates as

$$W_0 = \frac{1}{4\pi}\int dt\,(1-\cos\theta)\partial_t\phi = \frac{1}{4\pi}\int d\phi\,(1-\cos\theta) = \frac{\Omega}{4\pi}, \tag{7.8}$$

where Ω is a solid angle encompassed by the trajectory of $\vec{n}(t)$ during time evolution. The first term in the action (7.7) has a form of $\int dt\, p_\phi\dot{\phi}$ and the second is a negative time integral of the Hamiltonian.

Although (7.8) has a nice geometrical meaning it is written in some particular coordinate system on two-dimensional sphere. It would be nice to have an expression for W_0 which is coordinate independent and explicitly $SU(2)$ invariant (with respect to rotations of $\vec{n}$). Such form, indeed exists

$$W_0 = \int_0^1 d\rho \int_0^\beta dt\, \frac{1}{8\pi}\epsilon^{\mu\nu}\vec{n}\cdot\left[\partial_\mu\vec{n}\times\partial_\nu\vec{n}\right]. \tag{7.9}$$

Here we assume periodic boundary conditions in time $\vec{n}(\beta) = \vec{n}(0)$, ρ is an auxiliary coordinate $\rho \in [0,1]$. $\vec{n}$-field is extended to $\vec{n}(t,\rho)$ in such a way that $\vec{n}(t,0) = (0,0,1)$ and $\vec{n}(t,1) = \vec{n}(t)$. Indices μ,ν take values t,ρ.

Wess-Zumino action (7.9) has a very special property. Although it is defined as an integral over two-dimensional disk parameterized by ρ and t

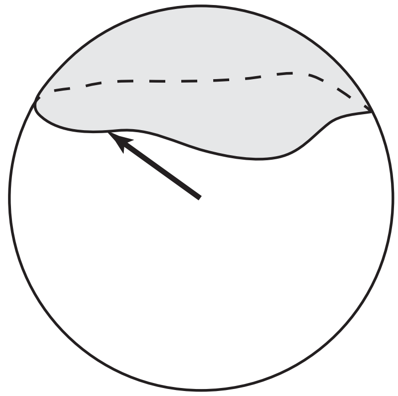


Fig. 7.1. The unit vector $\vec{n}(\tau)$ draws a closed line on the surface of a sphere with unit radius during its motion in imaginary time. Berry phase is proportional to the solid angle (shaded region) swept by the vector $\vec{n}(t)$. One can calculate this solid angle by extending $\vec{n}$ into the two-dimensional domain $\mathcal{B}$ as $\vec{n}(\rho, t)$ and calculating (7.9).

its variation depends only on the values of $\vec{n}$ on the boundary of the disk—physical time. Indeed one can check that

$$\begin{aligned} \delta W_0 &= \int_0^1 d\rho \int_0^\beta dt\, \frac{1}{4\pi}\epsilon^{\mu\nu}\vec{n}\cdot[\partial_\mu\delta\vec{n}\times\partial_\nu\vec{n}] \\ &= \int_0^1 d\rho \int_0^\beta dt\, \partial_\mu\left\{\frac{1}{4\pi}\epsilon^{\mu\nu}\vec{n}\cdot[\delta\vec{n}\times\partial_\nu\vec{n}]\right\} \\ &= \frac{1}{4\pi}\int_0^\beta dt\, \delta\vec{n}\left[\dot{\vec{n}}\times\vec{n}\right], \end{aligned} \tag{7.10}$$

where we used that $\delta\vec{n}\,[\partial_\mu\delta\vec{n}\times\partial_\nu\vec{n}] = 0$ because all three vectors $\delta\vec{n}$, $\partial_\mu\vec{n}$, and $\partial_\nu\vec{n}$ lie in the same plane (tangent to the two-dimensional sphere $\vec{n}^2 = 1$. Due to this property classical equation of motion does not depend on the arbitrary extension of $\vec{n}$ to $\rho \neq 1$.

In quantum physics, however, not only the variation δW_0 but the weight $e^{2\pi i W_0}$ should not depend on unphysical configuration $\vec{n}(t,\rho)$ but only on $\vec{n}(t,\rho = 1)$. To see that this is indeed so we consider the configuration $\vec{n}(t,\rho)$ as a mapping from two-dimensional disk $(t,\rho) \in \mathcal{B}_+$ into the two-dimensional sphere $\vec{n} \in S^2$. Suppose now that we use another extension $\vec{n}'(t,\rho)$ and represent it as a mapping of another disk $\mathcal{B}_-$ with the same boundary (physical time) into S^2. We have

$$W_0[\vec{n}] - W_0[\vec{n}'] = \int_{\mathcal{B}_+} d^2x\, \frac{1}{8\pi}\epsilon^{\mu\nu}\vec{n}\cdot[\partial_\mu\vec{n}\times\partial_\nu\vec{n}]$$

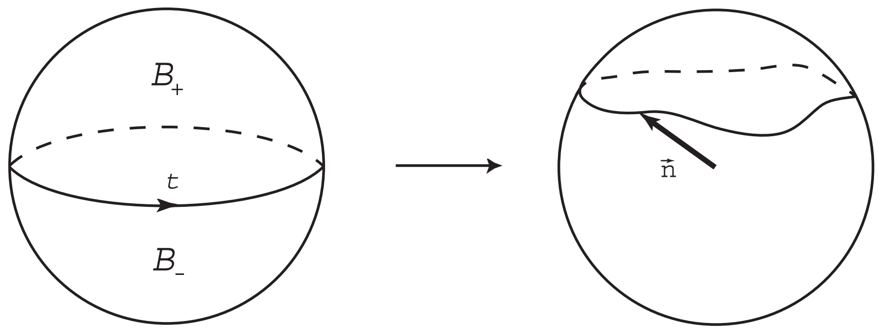


Fig. 7.2. Two extensions $\vec{n}(t,\rho)$ and $\vec{n}'(t,\rho)$ define a mapping $S^2 \to S^2$. The difference $W_0[\vec{n}] - W_0[\vec{n}']$ gives a winding number of this mapping.

$$
\begin{aligned}
&- \int_{\mathcal{B}_-} d^2x\, \frac{1}{8\pi}\epsilon^{\mu\nu}\vec{n}' \cdot [\partial_\mu \vec{n}' \times \partial_\nu \vec{n}'] \\
&= \int_{S^2=\mathcal{B}_+\cup\mathcal{B}_-} d^2x\, \frac{1}{8\pi}\epsilon^{\mu\nu}\vec{n} \cdot [\partial_\mu \vec{n} \times \partial_\nu \vec{n}] = k, \quad (7.11)
\end{aligned}
$$

where we changed orientation of $\mathcal{B}_-$ and considered $\mathcal{B}_\pm$ as an upper (lower) part of some two-dimensional sphere (see Fig.7.2). One can recognize the last integral [9] as a winding number k of the first sphere ($\mathcal{B}_+ \cup \mathcal{B}_-$ around the second $\vec{n} \in S^2$. This number is always integer proving that $e^{2\pi i W_0}$ does not depend on the particular way of an extension $\vec{n}(t,\rho)$. We notice here that in general the topological term W_0 can appear in the action only with the coefficient which is a multiple of $2\pi i$. Otherwise, it depends on the unphysical values of $\vec{n}(t,\rho)$ and is not defined. Such a term is called† "Wess-Zumino term" or "WZ term" by names of Wess and Zumino who discovered similar term first in the context of four-dimensional quantum field theories[**?**]. If Wess-Zumino term is present with some coupling constant g so that the weight in partition function is proportional to $e^{2\pi i g W_0}$ we immediately conclude that g must be integer. This phenomenon is called "topological quantization" of physical constant g and is a very important consequence of Wess-Zumino term.

To obtain the equations of motion from (7.7,7.9) we use (7.10) and introduce Lagrange multiplier λ to enforce constraint $\vec{n}^2 = 1$. Then we obtain

† It is also often called WZW or Wess-Zumino-Witten or even WZWN or Wess-Zumino-Novikov-Witten term to honor E. Witten[12, 13] and S.P. Novikov[14].

for the variation of the action

$$\delta_n\left(S[\vec{n}]+\lambda(\vec{n}^2-1)\right)=-4\pi S\frac{1}{4\pi}\left[\dot{\vec{n}}\times\vec{n}\right]+S\vec{h}+2\lambda\vec{n}=0. \tag{7.12}$$

Vector-multiplying (7.12) from the right by $\vec{n}$ we arrive to (7.5).

In this simplified treatment we just found some classical action which reproduces the classical limit of operator equations of motion (7.4). One can proceed more formally starting with commutation relations (7.1) and quantum Hamiltonian (7.3) and derive the classical action (7.7) using, e.g., coherent states method[23].

The purpose of this exercise was to illustrate that the Wess-Zumino term W_0 summarizes at the classical level the commutation relations (7.1). One can also show that reversely the path integral quantization of (7.12) produces the commutation relations (7.1).

7.2 Fermionic model

In this section we use a very simple quantum mechanical example to show how topological terms are generated when one passes from microscopic theory to an effective description. Generally, in condensed matter physics we are dealing with some system of electrons interacting with each other as well as with other degrees of freedom such as lattice. Let us assume that at some low energy scale we reduced our problem to a fermions interacting to some bosonic field. The bosonic field may originate both from the collective behavior of electrons, e.g., superconducting order parameter, and from independent degrees of freedom, e.g., from the lattice. For our illustrative example we consider

$$S=\int dt\,\psi^\dagger\left[i\partial_t+m\vec{n}\vec{\sigma}\right]\psi, \tag{7.13}$$

where m is a coupling constant, $\psi=(\psi_1,\psi_2)^t$ is a spinor, and $\vec{\sigma}$ is a triplet of Pauli matrices. In this case fermions are represented by just one spinor and bosonic field by a single unit vector $\vec{n}=(n_1,n_2,n_3)$, $\vec{n}\in S^2$. The latter means that $\vec{n}$ takes its values on a two-dimensional sphere, i.e., $\vec{n}^2=1$. This model can originate, e.g., from electrons interacting with a localized magnetic moment. Then coupling constant $m>0$ corresponds to a Hund's coupling between electrons (one electron for simplicity) and the direction $\vec{n}$ of a localized moment. Notice, that the more complete theory must have the bare action of a moment $\vec{n}$ added to a (7.13). We, however, are interested only in the action of $\vec{n}$ induced by an interaction with fermion.

For future convenience we will use a Euclidian formulation here and in the rest of the paper. It can be obtained by "Wick rotation" $t \to it$. A Euclidian action obtained from (7.13) is

$$S_E = \int dt \, \psi^\dagger \left[\partial_t - m\vec{n}\vec{\sigma}\right] \psi. \tag{7.14}$$

7.2.1 Effective action by chiral rotation trick

We consider partition function

$$Z = \int \mathcal{D}\psi \, \mathcal{D}\bar{\psi} \, \mathcal{D}\vec{n} \, e^{-S_E} = \int \mathcal{D}\vec{n} \, e^{-S_{eff}}, \tag{7.15}$$

where the last equality is a definition of an effective action

$$S_{eff} = -\ln \int \mathcal{D}\psi \, \mathcal{D}\bar{\psi} \, e^{-S_E} = -\ln \det D, \tag{7.16}$$

where we defined an operator $D \equiv \partial_t - m\vec{n}\vec{\sigma}$. To calculate the logarithm of the determinant we use "chiral rotation". Namely, we introduce the matrix field $U(t) \in SU(2)$ such that $U^\dagger \vec{n}\vec{\sigma} U = \sigma^3$ so that

$$\tilde{D} = U^\dagger D U = \partial_t - i\hat{a} - m\sigma^3, \tag{7.17}$$

with

$$\hat{a} \equiv U^\dagger i\partial_t U. \tag{7.18}$$

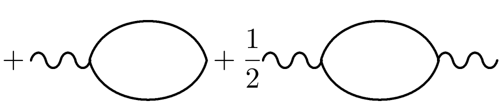
Then we write†

$$S_{eff} = -\ln \det D = -\ln \det \tilde{D} = -\mathrm{Tr} \, \ln \tilde{D}. \tag{7.19}$$

Let us now write $\tilde{D} = G_0^{-1}(1 - G_0 i\hat{a})$ and expand

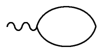
$$\begin{aligned} S_{eff} = -\mathrm{Tr} \, \ln \tilde{D} \;&=\; \mathrm{Tr} \left[\ln G_0 + G_0 i\hat{a} + \frac{1}{2}(G_0 i\hat{a})^2 + \ldots\right] \\ &=\; S^{(0)} + S^{(1)} + S^{(2)} + \ldots. \end{aligned} \tag{7.20}$$

Here $G_0 = (\partial_t - m\sigma^3)^{-1}$ and the expansion (7.20) has the following diagrammatic representation

$S_{eff} = const +$  $+ \ldots.$ (7.21)

† Notice that the second equality in (7.19) is the common source of miscalculated topological terms. Quantum anomalies might be present making chiral rotation technique inapplicable. In this case this is a legitimate procedure but see the discussion in Sec.??.

Here $S^{(0)}$ is an (infinite) constant which does not depend on $\hat{a}$. The first order term is given by

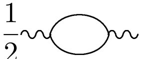
$$S^{(1)} =$$

$$= \mathrm{Tr}\,[G_0 i\hat{a}] = \int \frac{d\omega}{2\pi}\,\mathrm{tr}\left[\frac{1}{-i\omega - m\sigma^3} i\hat{a}_{\omega=0}\right]. \tag{7.22}$$

Here $\hat{a}_{\omega=0} = \int dt\,\hat{a}(t)$ and tr is taken over sigma-matrices. Closing the integral in an upper complex ω-plane we obtain

$$S^{(1)} = \int \frac{d\omega}{2\pi}\,\mathrm{tr}\left[\frac{1}{-i\omega - m\sigma^3} i\hat{a}_{\omega=0}\right] = -ia^3_{\omega=0} = -i\int dt\, a^3(t), \tag{7.23}$$

where only the term containing a^3 ($\hat{a} = a^k\sigma^k$, $k = 1,2,3$) does not vanish when trace over Pauli matrices is taken.†

We may proceed and obtain for the second term of an expansion

$$S^{(2)} = \frac{1}{2}$$

$$= \frac{1}{2}\mathrm{Tr}\,[G_0 i\hat{a} G_0 i\hat{a}]$$

$$\begin{aligned}
&= \frac{1}{2}\int \frac{d\Omega}{2\pi}\int \frac{d\omega}{2\pi}\,\mathrm{tr}\left[\frac{1}{-i\omega - m\sigma^3} i\hat{a}_{-\Omega}\frac{1}{-i(\omega+\Omega) - m\sigma^3} i\hat{a}_{\Omega}\right] \\
&= \frac{1}{8m}\int \frac{d\Omega}{2\pi}\,\mathrm{tr}\left[\hat{a}_{-\Omega}\hat{a}_{\Omega} - \sigma^3\hat{a}_{-\Omega}\sigma^3\hat{a}_{\Omega}\right] + o\left(\frac{1}{m}\right) \\
&= \frac{1}{2m}\int dt\,\left[(a^1)^2 + (a^2)^2\right] + o\left(\frac{1}{m}\right).
\end{aligned} \tag{7.24}$$

We neglected here the terms of higher order in $1/m$. Therefore, for an effective action we obtain up to the terms of the order of $1/m$ and omitting constant

$$S_{eff} = S^{(1)} + S^{(2)} = -i\int dt\, a^3(t) + \frac{1}{2m}\int dt\,\left[(a^1)^2 + (a^2)^2\right]. \tag{7.25}$$

The effective action (7.25) is expressed in terms of an auxiliary gauge field $\hat{a}$. However, one should be able to re-express it in terms of physical variable $\vec{n}$ as it was defined by (7.16) which contains only $\vec{n}$. Let us start with the second term. Using an explicit relation $\vec{n}\vec{\sigma} = U\sigma^3U^\dagger$ and the definition (7.18) one can easily check that $(\partial_t\vec{n})^2 = 4\left[(a^1)^2 + (a^2)^2\right]$ and the last term of (7.25) indeed can be expressed in terms of $\vec{n}$ as

$$S^{(2)} = \frac{1}{2m}\int dt\,\left[(a^1)^2 + (a^2)^2\right] = \frac{1}{8m}\int dt\,(\partial_t\vec{n})^2. \tag{7.26}$$

Things are more complicated with $S^{(1)}$. The gauge field is defined as (7.18) with matrix U defined implicitly by $\vec{n}\vec{\sigma} = U\sigma^3U^\dagger$. One can see from the

† We notice that ω-integral in (7.23) is formally diverging. However, being regularized it becomes the number of fermions in the system. In our model we have exactly one fermion and regularization procedure in this case is just the closing of the contour of an integration in an upper complex plane.

latter expression that the definition of U is ambiguous. Indeed, one can make a "gauge transformation"

$$U \to U e^{i\sigma^3 \psi} \tag{7.27}$$

with $\psi(t)$ any function of t without changing $\vec{n}$

$$\vec{n} \to \vec{n}. \tag{7.28}$$

Under this transformation the gauge field is transformed as $\hat{a} \to e^{-i\sigma^3\psi}\hat{a}e^{i\sigma^3\psi} - \sigma^3\partial_t\psi$ or

$$a^3 \;\to\; a^3 - \partial_t\psi, \tag{7.29}$$
$$a^1 \;\to\; a^1 \cos 2\psi - a^2 \sin 2\psi, \tag{7.30}$$
$$a^2 \;\to\; a^1 \sin 2\psi + a^2 \cos 2\psi. \tag{7.31}$$

Therefore, $S^{(1)} \to S^{(1)} + i \int dt\, \partial_t\psi$ and we notice that $S^{(1)}$ transforms non-trivially† under the change of U and therefore, can not be expressed as a simple time integral over the function which depends on $\vec{n}$ only. One might question the validity of our derivation because it seems that $S^{(1)}$ defined by (7.23) is not invariant under the transformation (7.27) but we notice that $S^{(1)}$ changes only by the integral of a full time derivative. Moreover, if we require periodicity in time, i.e., time changes from 0 to β and $\psi(\beta) = \psi(0) + 2\pi n$ with an integer n, then $S^{(1)} \to S^{(1)} + 2\pi i n$ and "Boltzmann" weight $e^{-S^{(1)}}$ is invariant under (7.27). Therefore, the contribution to the partition function from the $S^{(1)}$ term depends only on the physical variable $\vec{n}$. To understand what is going on let us calculate $S^{(1)}$ explicitly. We parametrize $\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Then, the most general choice of U is

$$U = \begin{pmatrix} \cos\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} e^{i\sigma^3\psi},$$

where $\psi(t)$ is an arbitrary function of t with $\psi(\beta) = \psi(0) + 2\pi n$. It is straightforward to calculate

$$a_3 = -\frac{1-\cos\theta}{2}\partial_t\phi - \partial_t\psi.$$

We see that the last term can be discarded by reasons given above and we have

$$S^{(1)} = 2\pi i W_0, \tag{7.32}$$

with W_0 defined in (7.8).

† Notice that (7.26) does not transform under this gauge transformation.

Combining (7.26) and (7.32) together we obtain

$$S_{eff} = 2\pi i W_0 + \frac{1}{8m}\int dt\,(\partial_t \vec{n})^2 + o\left(\frac{1}{m}\right). \tag{7.33}$$

In the case where N species of fermions coupled to the same $\vec{n}$ field are present, one obtains an overall factor N in effective action, i.e., $S_{eff} \to NS_{eff}$.

7.2.2 Topological term

Let us notice that the first term $2\pi i W_0$ of gradient expansion (7.33) is very different from, say, the second one in the following respects

(i) It is imaginary.
(ii) It does not depend on m.†
(iii) It does not change under reparameterezation of time $t \to f(t)$. In particular, it is scale invariant and does not change when $t \to \lambda t$.

The property (iii) makes it natural to call the term $2\pi i W_0$ topological as it does not depend on time scales but only on the trajectory of $\vec{n}(t)$. In fact, properties (i), (ii) are consequences of (iii) as it will be shown below are the general properties of all topological terms.

7.2.3 Path integral representation of quantum spin

Before going to the next section let us consider some application of derived topological term. We generalize our model slightly so that in (7.14) ψ denotes N species of fermions which are all coupled to the same bosonic field $\vec{n}$. We consider the special limit $m \to \infty$ of the model (7.14). The Hamiltonian of the model $-m\vec{n}\psi^\dagger\vec{\sigma}\psi$ in this limit forces all spins of ψ particles to be aligned along $\vec{n}$. Therefore, we expect that in this limit after an integration over fermions we will obtain an effective action of a quantum spin $S = N/2$ written in terms of the direction $\vec{n}$ of its quantization axis. Multiplying (7.33) by the number of fermion species N and taking limit $m \to \infty$ we obtain

$$S = 2\pi i N W_0. \tag{7.34}$$

One can show that upon quantization‡ the components of $\vec{n}$ become the components of the quantum spin $S = N/2$ so that $n_a \to \hat{S}_a/S$. The action

† If we allow m to be negative this term becomes $2\pi i\,\mathrm{sgn}\,m W_0$ and depends only on the sign of m, not its magnitude.

‡ The easy way to show that we are dealing with the spin is to add coupling to an external magnetic field $-\int dt\, S\vec{h}\vec{n}$ to (7.34) and write down the classical equation of motion for $\vec{n}$ using

(7.34) is explicitly $SU(2)$ invariant and is well-defined for integer $2S = N$, i.e. spin can only be integer or half-integer. Therefore, in path integral representation the quantization of spin is a consequence of the Wess-Zumino term in the action of the spin.

7.3 Derivation of a WZ term from fermionic model without chiral rotation

Here we give an alternative derivation of an effective action (7.33) from (7.16) which does not use chiral rotation trick.

Consider

$$Z = \int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, \mathcal{D}\vec{n}\, e^{-S}, \tag{7.35}$$

where

$$S = \int_0^T dt\, \bar{\psi}\,(i\partial_t - iM\vec{n}\vec{\tau})\,\psi. \tag{7.36}$$

Here $\psi = (\psi_1, \psi_2)$ is a Grassman spinor representing spin 1/2 fermion, $\vec{\tau}$ - Pauli matrices acting on spinor indices of ψ and $\vec{n}^2 = 1$ is a unit, three-component vector coupled to the spin of fermion $\bar{\psi}\vec{\tau}\psi$ with the coupling constant M.

Integrating out fermions in (7.35) we obtain

$$Z = \int \mathcal{D}\vec{n}\, e^{-S_{\text{eff}}(\vec{n})}, \tag{7.37}$$

where

$$S_{\text{eff}}(\vec{n}) = -\ln\det\,(i\partial_t - iM\vec{n}\vec{\tau})\,. \tag{7.38}$$

Let us denote $D = i\partial_t - iM\vec{n}\vec{\tau}$ and $D^\dagger = i\partial_t + iM\vec{n}\vec{\tau}$. We calculate the variation of the effective action

$$\begin{aligned} \delta S_{\text{eff}} &= -\text{Tr}\left\{\delta D D^{-1}\right\} = -\text{Tr}\left\{\delta D D^\dagger (D D^\dagger)^{-1}\right\} \\ &= iM\text{Tr}\left\{\delta\vec{n}\vec{\tau}(i\partial_t + iM\vec{n}\vec{\tau})(-\partial_t^2 + M^2 - M\dot{\vec{n}}\vec{\tau})^{-1}\right\} \end{aligned} \tag{7.39}$$

Expanding the fraction in $\dot{\vec{n}}$, calculating the trace and keeping only lowest orders in $\dot{\vec{n}}/M$ we obtain:

$$\delta S_{\text{eff}} = \int dt\, \left\{\frac{1}{4M}\delta\dot{\vec{n}}\,\dot{\vec{n}} - \frac{i}{2}\delta\vec{n}\cdot\left[\vec{n}\times\dot{\vec{n}}\right]\right\}. \tag{7.40}$$

(7.10) and constraint $\vec{n}^2 = 1$. We obtain $\dot{\vec{n}} = [\vec{n}\times\vec{h}]$ where we assumed $S = N/2$ and changed to the real time $t \to it$. The obtained equation is indeed the classical equation of spin precession.

Restoring the effective action from its variation we have:

$$S_{\text{eff}} = \int_0^T dt\, \frac{1}{8M}\dot{\vec{n}}^2 - 2\pi i W_0, \tag{7.41}$$

where *Wess-Zumino action*

$$W_0 = \int_0^1 d\rho \int_0^T dt\, \frac{1}{8\pi}\epsilon^{\mu\nu}\vec{n}\cdot\left[\partial_\mu\vec{n}\times\partial_\nu\vec{n}\right]. \tag{7.42}$$

Here ρ is an auxilliary coordinate $\rho \in [0,1]$. $\vec{n}$-field is extended to $\vec{n}(t,\rho)$ in such a way that $\vec{n}(t,0) = (0,0,1)$ and $\vec{n}(t,1) = \vec{n}(t)$. Indices μ,ν take values t,ρ.

Wess-Zumino action (7.42) has a very special property. Although it is defined as an integral over two-dimensional disk parameterized by ρ and t its variation depends only on the values of $\vec{n}$ on the boundary of the disk—physical time.

7.4 Quantum spin as a particle moving in the field of Dirac monopole

If we interpret the $\vec{n}$ as a position of particle on a unit two-dimensional sphere, this is exactly the phase picked by particle moving in the field of Dirac monopole of magnetic charge $2S$.

One could have started with the problem of particle of the mass m moving in the field of magnetic monopole of charge $2S$. Then the ground state is $2S$-degenerate and is separated by the gap $\sim 1/m$ from the rest of the spectrum. In the limit $m \to 0$ only the ground state is left and we obtain a quantum spin problem in an approach analagous to the plane rotator from Sec.2.4.

7.5 Reduction of a WZ term to a theta-term

Let us consider the value of (7.8) assuming that the polar angle is kept constant $\theta(\tau) = \theta_0$. Then (7.8) becomes

$$W_0 = \frac{1-\cos\theta_0}{2}\int_0^\beta \frac{d\tau}{2\pi}\,\partial_\tau\phi \tag{7.43}$$

and we recognize (7.32) with (7.43) as theta-term (2.11) corresponding to the particle on a ring with the flux through the ring given by

$$A = \frac{1-\cos\theta_0}{2}. \tag{7.44}$$

In particular, $\theta_0 = \pi/2$ the topological term in the action of a particle on a ring in magnetic field $A = 1/2$.

7.6 Properties of WZ terms

WZ terms

(i) are metric independent
(ii) are imaginary in Euclidian formulation
(iii) do not contribute to stress-energy tensor (and to Hamiltonian).
(iv) do not depend on m – the scale, below which an effective action is valid (but do depend on $\mathrm{sgn}\,(m)$)
(v) are antisymmetric in derivatives with respect to different space-time coordinates (contain $\epsilon^{\mu\nu\lambda\ldots}$)
(vi) are written as integrals of (D+1)-forms over auxiliary $(D+1)$-dimensional space - disk D^{D+1} such that $\partial D^{D+1} = S^D$ - compactified space-time
(vii) are multi-valued functionals. Multi-valuedness results in quantization of coupling constants (coefficients in front of WZ terms)
(viii) do change equations of motion by changing commutation relation between fields (Poisson's brackets) not by changing Hamiltonian
(ix) might lead to massless excitations with "half-integer spin"
(x) describe boundary theories of models with θ-terms
(xi) being combined (see the spin chains chapter) produce θ-terms
(xii) can be calculated by gradient expansion of the variation of fermionic determinants
(xiii) produce θ terms as a reduction of target space.

Among the listed properties the first five (i)-(v) are the properties of all topological terms while the others are more specific to WZ terms.

7.7 Exercises

Exercise 1: WZW in $0+1$, preliminaries

Consider a three-dimensional unit vector field $\vec{n}(x,y)$ ($\vec{n} \in S^2$) defined on a two-dimensional disk D. Define

$$W_0 = \int_D d^2x\, \frac{1}{8\pi}\epsilon^{\mu\nu}\vec{n}[\partial_\mu\vec{n} \times \partial_\nu\vec{n}] = \int_D \frac{1}{16\pi i}\mathrm{tr}\,[\hat{n}d\hat{n}d\hat{n}], \tag{7.45}$$

where the latter expression is written in terms of differential forms and $\hat{n} = \vec{n}\cdot\vec{\sigma}$.

a) Calculate the variation of W_0 with respect to $\vec{n}$. Show that the integral becomes the integral over disk D of the complete divergence (of the exact form).

b) Parametrize the boundary ∂D of the disk by parameter t, apply Gauss-Stokes theorem and express the result of the variation using only the values of $\vec{n}(t)$ at the boundary.

We showed that the variation of W depends only on the boundary values of $\vec{n}$-field.

Exercise 2: WZW in $0+1$*, definition*

Assume that we are given the time evolution of $\vec{n}(t)$ field ($\vec{n} \in S^2$). We also assume that time can be compactified, i.e. $\vec{n}(t = \beta) = \vec{n}(t = 0)$. Consider the two-dimensional disk D which boundary ∂D is parametrized by time $t \in [0, \beta]$. The WZW term is defined by

$$S_{WZW} = i4\pi S\, W_0[\vec{n}], \tag{7.46}$$

where S is some constant, W_0 is given by Eq. (7.45), and $\vec{n}(x, y)$ is some arbitrary smooth extension of $\vec{n}(t)$ from the boundary to an interior of the disk.

Let us show that the WZW term is well defined and (almost) does not depend on the extension of $\vec{n}(t)$ to the interior of D.

Consider two different extensions $\vec{n}^{(1)}(x, y)$ and $\vec{n}^{(2)}(x, y)$ of the same $\vec{n}(t)$ and corresponding values $W_0^{(1)}$ and $W_0^{(2)}$ of the functional W_0. Show that the difference $W_0^{(1)} - W^{(2)}$ is an integer number - the degree Q of mapping $S^2 \to S^2$. The second S^2 here is a target space of $\vec{n}$. How did the first S^2 appear?

We see that $S_{WZW}[\vec{n}(t)]$ is a multi-valued functional which depends on the extension of $\vec{n}$ to the disk D. However, the weight in partition function is given by $e^{-S_{WZW}}$ and can be made single-valued functional if the coupling constant S is "quantized". Namely, if $2S \in \mathbf{Z}$ (S - half-integer number) the $e^{-S_{WZW}}$ is a well-defined single-valued functional.

Exercise 3: WZW in $0+1$*, spin precession*

Let us consider the quantum-mechanical action of the unit vector $\vec{n}(t)$ with the (Euclidian) action

$$S_h = S_{WZW}[\vec{n}(t)] - S \int dt\, \vec{h} \cdot \vec{n}(t), \tag{7.47}$$

where W is given by (7.45) and $\vec{h}$ is some constant three-component vector (magnetic field).

Find the classical equation of motion for $\vec{n}(t)$ from the variational principle $\delta S_h = 0$. Remember that one has a constraint $\vec{n}^2 = 1$ which can be taken into account using, e.g., Lagrange multiplier trick.

The obtained expression is the equation of spin precession and S_{WZW} is a proper, explicitly $SU(2)$ invariant action for the free spin S.

Exercise 4: WZW in $0+1$, quantization

Show that the classical equations of motion obtained from S_h correspond to Heisenberg equations (in real time) $\partial_t \hat{\vec{S}} = i\left[H, \hat{\vec{S}}\right]$ for the quantum spin operator $\hat{\vec{S}}$

$$\left[S^a, S^b\right] = i\epsilon^{abc} S^c \tag{7.48}$$

obtained from the Hamiltonian of a spin in magnetic field

$$H = -\vec{h} \cdot \hat{\vec{S}}. \tag{7.49}$$

Obtain the commutation relations of quantum spin (7.48) from the topological part S_{WZW}. Notice that this topological action is linear in time derivative and, therefore, does not contribute to the Hamiltonian. Nevertheless, it defines commutation relations between components of the spin operator.

Hint: You can either use local coordinate representation of the unit vector in terms of spherical angles $\vec{n} = (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)$ or use the general formalism of obtaining Poisson bracket from the symplectic form given in S_{WZW}.

Exercise 5: Reduction of WZW to the theta-term in $0+1$

Let us assume that the field $\vec{n}(t)$ is constrained so that it takes values on a circle given in spherical coordinates by $\theta = \theta_0 = const$. Find the value of the topological term S_{WZW} on such configurations (notice that this constraint is not applicable in the interior of the disk D, only at its physical boundary). Show that the obtained topological term is a theta-term in $0+1$ corresponding to $S^1 \to S^1$.

What is the value of the coefficient in front of that topological term? What is the value of corresponding "magnetic flux" through a ring? For $S = 1/2$ which reduction (value of θ_0) corresponds to the half of the flux quantum?

Exercise 6: Particle in a field of Dirac monopole

Exercise 7: Berry's phase

Exercise 8: Coherent states

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