8

Brief introduction to differential forms

> "Hamiltonian mechanics cannot be understood without differential forms"
>
> V. I. Arnold

In this chapter we give a very brief introduction to differential forms following the chapter 7 of Ref.[26]. We refer the reader to [26, 9, 8] for more detailed (and more precise) introductions.

8.1 Exterior forms

8.1.1 Definitions

Let R^n be an n-dimensional real vector space.

Definition An exterior algebraic form of degree k (or k-form) is a function of k vectors which is k-linear and antisymmetric.

Namely:

$$\omega(\lambda_1\xi_1' + \lambda_2\xi_1'', \xi_2, \ldots, \xi_k) = \lambda_1\omega(\xi_1', \xi_2, \ldots, \xi_k) + \lambda_2\omega(\xi_1'', \xi_2, \ldots, \xi_k),$$

and

$$\omega(\xi_{i_1}, \ldots, \xi_{i_k}) = (-1)^\nu\omega(\xi_1, \ldots, \xi_k)$$

with $(-1)^\nu = 1\,(-1)$ if permutation $(i_1, \ldots, i_k)$ is even (odd). Here $\xi_i \in R^n$ and $\lambda_i \in R$.

The set of all k-forms in R^n form a real vector space with

$$(\omega_1 + \omega_2)(\xi) = \omega_1(\xi) + \omega_2(\xi), \qquad \xi = \{\xi_1, \ldots, \xi_k\}$$

and

$$(\lambda\omega)(\xi) = \lambda\omega(\xi).$$

Let us suppose that we have a system of linear coordinates $x_1, \ldots, x_n$ on R^n. We can think of these coordinates as of 1-forms so that $x_i(\xi) = \xi_i$ - the i-th coordinate of vector ξ. These coordinates form a basis of 1-forms in the

1-form vector space (which is also called the dual space $(R^n)^*$). Any 1-form can be written as a linear combination of basis 1-forms

$$\omega_1 = a_1 x_1 + \ldots + a_n x_n.$$

8.1.2 Exterior products

Definition An exterior product of k 1-forms $\omega_1, \omega_2, \ldots, \omega_k$ is a k-form defined by

$$(\omega_1 \wedge \ldots \wedge \omega_k)(\xi_1, \ldots, \xi_k) = \det \omega_i(\xi_j).$$

Exterior products of basis 1-forms $x_{i_1} \wedge \ldots \wedge x_{i_k}$ with $i_1 < \ldots < i_k$ form a basis in the space of k-forms. The dimension of the latter space is obviously C_n^k. A general k-form can be written as

$$\omega^k = \sum_{1 \le i_1 < \ldots < i_k \le n} a_{i_1 \ldots i_k} x_{i_1} \wedge \ldots \wedge x_{i_k},$$

where $a_{i_1 \ldots i_k}$ are real numbers.

Definition The exterior product $\omega^k \wedge \omega^l$ of a k-form with l-form on R^n is the $k + l$-form on R^n defined as

$$(\omega^k \wedge \omega^l)(\xi_1, \ldots, \xi_{k+l}) = \sum (-1)^\nu \omega^k(\xi_{i_1}, \ldots, \xi_{i_k}) \omega^l(\xi_{j_1}, \ldots, \xi_{j_l}),$$

where $i_1 < \ldots < i_k$ and $j_1, \ldots, j_l$ and the sum is taken over all permutations $(i_1, \ldots, i_k, j_1, \ldots, j_l)$ with $(-1)^\nu$ being $+1(-1)$ for even (odd) permutations.

One can check that this definition is consistent with the exterior product of 1-forms defined above and that the exterior product is distributive, associative, and skew-commutative. The latter means $\omega^k \wedge \omega^l = (-1)^{kl} \omega^l \wedge \omega^k$.

If ω is a 1-form (or a form of an odd degree) one can easily show that $\omega \wedge \omega = 0$.

8.1.3 Behavior under mappings

Let $f : R^m \to R^n$ be a linear map, and ω^k an exterior k-form on R^n. Then we can define a k-form $(f^* \omega^k)$ on R^m by

$$(f^* \omega^k)(\xi_1, \ldots, \xi_k) = \omega^k(f\xi_1, \ldots, f\xi_k).$$

Notice that the obtained mapping of forms f^* acts in "opposite" direction to f. Namely, $f^* : \Omega_k(R^n) \to \Omega_k(R^m)$, where $\Omega_k(R^n)$ is a vector space of k-forms on R^n.

8.2 Differential forms

Definition A differential k-form $\omega^k|_x$ at a point x of a manifold M is an exterior k-form on the tangent space TM_x to M at x, i.e., a k-linear skew-symmetric function of k vectors $\xi_1, \ldots, \xi_k$ tangent to M at x.

If such a form is given at every point x of M and if it is differentiable, we say that we are given a k-form ω^k on the manifold M.

We introduce the coordinate basis 1-forms dx_i with $i = 1, \ldots, n$ (n-dimension of M). The notation dx_i instead of x_i used for exterior forms emphasizes that these basis forms act on TM_x at a given point x of M.

In the neighborhood of the point x one can always write the general differential k-form as

$$\omega^k = \sum_{i_1 < \ldots < i_k} a_{i_1 \ldots i_k}(x)\, dx_{i_1} \wedge \ldots \wedge dx_{i_k}, \tag{8.1}$$

where $a_{i_1 \ldots i_k}(x)$ are smooth functions of x.

A simple example of a differential 1-form is the differential of some scalar function $f(x)$ defined on the manifold M. We introduce

$$df = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}\Big|_x dx_k,$$

where dx_k are basis differential 1-forms. The value of this 1-form on a vector $\xi \in TM_x$ is given by

$$df(\xi) = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}\Big|_x \xi_k.$$

8.2.1 Behavior under mappings

Let $f : M \to N$ be a differentiable map of a smooth manifold M to a smooth manifold N, and let ω be a differential k-form on N. The mapping f induces the mapping $f_* : TM_x \to TM_{f(x)}$ of tangent spaces. The latter mapping f_* is called the differential of the map f. The mapping f_* is a mapping of linear spaces and gives rise to the mapping of forms defined on corresponding tangent spaces (see Sec. 8.1.3). As a result a well-defined differential k-form $(f^*\omega)$ exists on M:

$$(f^*\omega)(\xi_1, \ldots, \xi_k) = \omega(f_*\xi_1, \ldots, f_*\xi_k). \tag{8.2}$$

8.3 Integration of differential forms over chains

8.3.1 Integration of k-form over k-dimensional cell

Let D be a bounded convex polyhedron in R^k and $x_1, \ldots, x_k$ an oriented coordinate system on R^k. Any differential k-form on R^k can be written as $\omega^k = \phi(x)\, dx_1 \wedge \ldots \wedge dx_k$, where $\phi(x)$ is a differentiable function on R^k. We define the integral of the form ω^k over D as the integral of the function ϕ:

$$\int_D \omega^k = \int_D \phi(x)\, dx_1 dx_2 \ldots dx_k. \tag{8.3}$$

Let us define a k-dimensional cell σ of n-dimensional manifold M as a polyhedron D in R^k defined above with a differentiable map $f : D \to M$. One can think about σ as of "curvilinear polyhedron" - the image of D on M. If ω is a differentiable k-form on M we define the integral of a form ω over the cell σ as

$$\int_\sigma \omega = \int_D f^* \omega, \tag{8.4}$$

where f^* is a mapping of k-forms induced by f (see Sec. 8.2.1).

The cell σ inherits an orientation from the orientation of R^k. The k-dimensional cell which differs from σ only by the choice of orientation is called the negative of σ and is denoted by $-\sigma$ or by $(-1)\sigma$. One can show that under a change of orientation the integral changes sign:

$$\int_{-\sigma} \omega = -\int_\sigma \omega. \tag{8.5}$$

8.3.2 Chains and boundary operator

It is convenient to generalize our definition of the integral of a form over *cell* to the integral over *chain*.

Definition A chain of dimension k on a manifold M consists of a finite collection of k-dimensional oriented cells $\sigma_1, \ldots, \sigma_r$ in M and integers $m_1, \ldots, m_r$, called multiplicities. A chain is denoted

$$c_k = m_1\sigma_1 + \ldots + m_r\sigma_r. \tag{8.6}$$

One can introduce the structure of a commutative group on a set of k-chains on M with natural definitions of addition of chains $c_k + b_k$.

The *boundary* of a convex oriented k-polyhedron D on R^k is the $(k-1)$-chain ∂D on R^k defined as

$$\partial D = \sum_i \sigma_i,$$

where the cells σ_i the $(k-1)$-dimensional faces of D with orientations inherited from the orientation of R^k (see [26]). One can easily extend this definition to the definition of the boundary of a cell $\partial\sigma$ on M and then to the boundary of a chain (8.6) as

$$\partial c_k = m_1 \partial\sigma_1 + \ldots + m_r \partial\sigma_r. \tag{8.7}$$

∂c_k is a $(k-1)$-chain on M. Additionally, we define a 0-chain is a collection of points with multiplicities and the boundary of an oriented interval $\vec{AB}$ is $B - A$. The boundary of a point is empty.

One can show that the boundary of the boundary of the cell is zero $\partial(\partial\sigma) = 0$ and therefore

$$\partial(\partial c_k) = 0$$

for any k-chain c_k. We write this property as

$$\partial\partial = 0. \tag{8.8}$$

8.3.3 Integration of k-form over k-chains

An integral of a k-form over k-chain (8.6) is defined as

$$\int_{c_k} \omega^k = \sum m_i \int_{\sigma_i} \omega^k. \tag{8.9}$$

8.4 Exterior differentiation and Stokes formula

An exterior derivative of the differential k-form ω^k is a $(k+1)$-form $\Omega = d\omega^k$. One can define it using a choice of coordinates on M as

$$\Omega = d\omega^k = \sum da_{i_1 \ldots i_k} \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k} \tag{8.10}$$

if ω^k is given by (8.1). Here da is a 1-form of a differential of a function $a(x)$.

One can show that the definition (8.10) does not actually depend on the choice of coordinates (see [26] for more rigorous discussion).

We think of the 1-form given by the differential of a scalar function (see Sec. 8.2) as of an external derivative of 0-form (scalar function).

It is easy to see from the definition (8.10) that

$$d(dx_i) = 0.$$

Using this property one can show that generally

$$d(d\omega) = 0,$$

where ω is a general differential k-form. We write the latter property as

$$dd = 0. \tag{8.11}$$

One can easily prove the formula for differentiating an exterior product of forms

$$d(\omega^k \wedge \omega^l) = d\omega^k \wedge \omega^l + (-1)^k \omega^k \wedge d\omega^l. \tag{8.12}$$

If $f : M \to N$ is a smooth map and ω is a k-form on N we have

$$f^*(d\omega) = d(f^*\omega). \tag{8.13}$$

8.4.1 Stokes' formula

One of the most important formulae in differential geometry is the Newton-Leibniz-Gauss-Green-Ostrogradskii-Stokes-Poincaré formula:

$$\int_{\partial c} \omega = \int_c d\omega, \tag{8.14}$$

where c is any $(k+1)$-chain on a manifold M and ω is any k-form on M.

The formula is proved similarly to Green's formula in calculus.

In particular case when the boundary $\partial c = 0$ we have $\int_c d\omega = 0$ (integration of a complete derivative over closed surface).

8.5 Homologies and Cohomologies

8.5.1 Closed and exact forms

Definition A differential form ω on a manifold M is *closed* if $d\omega = 0$.

In particular, on a 3d riemannian manifold we have $d\omega^2_{\vec{A}} = (\nabla \cdot \vec{A})\omega^3 = 0$ is equivalent to $(\nabla \cdot \vec{A}) = 0$, i.e. the corresponding vector field is divergenless (we used (8.30)).

Using Stokes formula (8.14) we have for a closed form

$$\int_{\partial c_{k+1}} \omega^k = 0, \qquad \text{if } d\omega^k = 0. \tag{8.15}$$

Definition A differential form ω on a manifold M is *exact* if $\omega = d\mu$, where μ is another differential form.

By (8.11) all exact forms are closed. However, there are closed forms which are not exact. For example on a circle S^1 parametrized by an angle $\phi \in [0, 2\pi]$ one can introduce the 1-form ω^1 defined by $\omega^1(\partial_t\gamma) = \partial_t\gamma$, where $\partial_t\gamma$ is the "velocity" along the path $\phi = \gamma(t)$ which belongs to the tangent space to S^1 at ϕ. The form is obviously closed ($d\omega^1$ is a 2-form which is automatically zero on S^1 - one-dimensional manifold). However,

$$\int_{S^1} \omega^1 = \int_0^T dt\, \partial_t \gamma = 2\pi.$$

Although $\partial S^1 = 0$, the integral is not zero, and, therefore, ω^1 is not exact.

We notice that locally the introduced 1-form can be written as $\omega^1 = d\phi$ but this form is not valid at $\phi = 0$. This is the general situation. According to *Poincarè's lemma* any closed form is locally exact. The existence of locally but not globally exact closed form is related to topological properties of M.

8.5.2 Cycles and boundaries

Definition A *cycle* on a manifold M is a chain whose boundary is equal to zero.

Using Stokes formula (8.14) we have

$$\int_{c_{k+1}} d\omega^k = 0, \qquad \text{if } \partial c_{k+1} = 0. \tag{8.16}$$

The chains which can be considered as boundaries of some other chains are called *boundaries*. By (8.8) all boundaries are cycles. However, generally, not all cycles are boundaries. The existence of cycles which are not boundaries is related to topological properties of M.

For example, let us take one of the cycles of torus. It is the cycle (its boundary is zero). However it does not bound any chain on torus.

8.5.3 Homologies and Cohomologies

The set of all k-forms on M is a vector space, the closed k-forms a subspace, and the differentials of $(k-1)$-forms (exact k-forms) a subspace of the subspace of closed forms. The quotient space

$$\frac{\text{(closed forms)}}{\text{(exact forms)}} = H^k(M, R) \tag{8.17}$$

is called the *k-th cohomology group* of the manifold M. An element of this group is a class of closed forms differing from one another only by an exact form.

For a circle S^1 we have $H^1(S^1, R) = R$.

The dimension of H^k is called the *k-th Betti number of M.* The first Betti number of S^1 is 1.

The cohomology groups of M are important *topological properties* of M.

Let us consider two *k-cycles* a and b such that their difference is a boundary of a $(k+1)$-chain, i.e., $a-b=\partial c_{k+1}$ (such cycles are called *homologous*). If $d\omega^k=0$ we have from (8.15)

$$\int_a \omega^k = \int_b \omega^k, \tag{8.18}$$

that is cycles can be replaces one by another.

The quotient group

$$\frac{\text{(cycles)}}{\text{(boundaries)}} = H_k(M) \tag{8.19}$$

is called the *k-th homology group of M*. An element of this group is a class of cycles homologous to one another. The rank of this group is also equal to k-th Betty number of M.

8.5.4 Homologies and homotopies

There are important relations between homology and homotopy groups of the topological space M (see e.g., [8]).

Let us suppose that $\pi_1(M)$ and $H_1(M)$ are fundamental group and first homology group of M respectively. Then, $H_1(M)=\pi_1(M)/[\pi_1,\pi_1]$, where $[\pi_1,\pi_1]$ is the commutator of the group $\pi_1(M)$.

In particular, if $\pi_1(M)$ is Abelian, then $\pi_1(M)=H_1(M)$.

For higher homotopy groups there is *Gurevich theorem*.

Gurevich theorem If $\pi_k(M)=0$ for all $k<n$. Then

$$\pi_n(M)=H_n(M). \tag{8.20}$$

Homology (and cohomology) groups are usually easier to calculate than homotopy groups.

8.6 Differential forms in physics

8.6.1 $U(1)$ Gauge field

Let us suppose that we have a gauge field $A_\mu(x)$ in euclidian space R^n. It is very useful to think of the gauge field as of 1-form

$$A = A_\nu\, dx^\nu. \tag{8.21}$$

The exterior differential of this form

$$\begin{aligned} F &= dA = \frac{\partial A_\nu}{\partial x^\mu} dx^\mu \wedge dx^\nu = \frac{1}{2}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) dx^\mu \wedge dx^\nu \\ &= \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu \end{aligned} \tag{8.22}$$

is a 2-form which has the field tensor $F_{\mu\nu}$ as coefficients.

If 1-form A is exact, i.e., $A = df$ with some scalar function f we call A *pure gauge*. In components $A_\mu = \partial_\mu f$. In this case $F = 0$ as $F = dA = d(df) = 0$ because of (8.11). On euclidian space R^n the inverse is also true. If $F = dA = 0$, i.e, A is *closed*, the form A is exact and there exists f such that $A = df$. This happens because R^n is a simply-connected space. Indeed, in a simply-connected space every 1-cycle c_1 is a boundary $c_1 = \partial c_2$ (homologous to zero). Therefore, having $F = 0$ we have by Stokes theorem $\int_{c_1} A = \int_{c_2} F = 0$.

Let us now assume that the 1-form A is given on a manifold M which is not simply connected $\pi_1(M) \neq 0$. Then, there are cycles which are not boundaries and the integral of A over such cycles might not vanish. It is possible to have, therefore, closed but not exact 1-forms A. In classical mechanics this is not essential as only 2-form F is physical but for closed 1-forms A we have $F = dA = 0$. In quantum mechanics, however the integral $\int_{c_1} A \neq 0$ is a phase which is picked up by quantum particle moving around the cycle c_1. One can observe the interference due to this phase even in cases where particle is never in the presence of "magnetic field" ($F = 0$ everywhere). This phenomenon is called *Aharonov-Bohm effect*.

The integral $\Phi = \int_{c_1} A \neq 0$ can be interpreted as a magnetic flux through the "holes" of a multiply connected space M. For example, the flux through the ring for the particle on the ring or two different fluxes through the non-trivial basic cycles of 2d torus etc.

8.6.2 Dirac monopole

We have seen that the 2-form F corresponding to the field tensor $F_{\mu\nu}$ is closed and this statement $dF = 0$ is equivalent to a pair of Maxwell's equations (Bianchi identity). On R^n this means that $F = dA$ where A is some 1-form given globally on R^n and F is necessarily exact. Let us consider now the closed 2-form F on a non-trivial manifold M (think of $M = S^2$ as an example). For a 2-dimensional surface c_2 with boundary $c_1 = \partial c_2$ we have $\int_{c_2} F$ - the flux through the surface c_2. According to Poincare's lemma we can write $F = dA$ with some 1-form A defined in the vicinity of c_2 (as it can be contracted to a point). Then we have $\int_{c_2} F = \int_{c_1} A$. If the integral of F over the other surface c_2' with the same boundary $\partial c_2' = c_1$ is different then the 1-form A can not be defined globally on M. Indeed

$$0 \neq \int_{c_2 - c_2'} F = \int_{c_1} (A - A').$$

It means that if the flux through some 2-cycle is not zero, the vector potential 1-form does not exist globally.

As an example we consider the particle moving on a sphere in a magnetic field of a point magnetic charge g placed in the center of the sphere $B = g/R^2$, where R is the radius of the sphere. The flux of magnetic field through the sphere $\int B\, d^2x = 4\pi g$ and considering c_2 and c_2' as northern and southern hemispheres we have

$$4\pi g \int_{c_2 - c_2'} F = \int_{c_1} (A - A') = \frac{1}{e}(\Delta\phi - \Delta\phi'),$$

where e is the electric charge of the particle and $\Delta\phi = e\int_{c_1} A$ is the phase picked up by quantum particle when moved around the equator c_1 of the sphere. For the consistency of quantum mechanics one should require that $\Delta\phi - \Delta\phi' = 2\pi \times \text{integer}$. Therefore, we have the famous Dirac monopole charge quantization condition

$$2ge = \text{integer}. \tag{8.23}$$

8.6.3 Theta and WZW terms for spheres

Let us consider the sphere S^n. We parameterize this manifold by "embedding coordinates" $\vec{\pi} = (\pi^1, \ldots, \pi^{n+1})$ with $\vec{\pi}^2 = 1$. We have $H^k(S^n, R) = R$ for $k = 0, n$ and 0 for $k \neq 0, n$. The normalized n-form from $H^n(S^n, R)$ is a volume form on S^n which can be explicitly written as

$$\omega^n = \frac{1}{n!\,\text{Area}\,(S^n)} \epsilon^{a_1 \ldots a_{n+1}}\, \pi^{a_{n+1}} d\pi^{a_1} \wedge \ldots \wedge d\pi^{a_n}. \tag{8.24}$$

Let us use this forms to write down the expressions for theta and WZW terms with spheres as target spaces.

8.6.3.1 Theta terms

Consider D-dimensional spacetime S^D and a target space $M = S^D$. We introduce

$$Q[\pi(x)] = \int_{S^D} \pi^* \omega^D, \tag{8.25}$$

where the integral is taken over the spacetime and $\pi^*\omega^D$ is the pullback of the volume form ω^D existing on the target space $M = S^D$. The value of (8.25) is the degree of mapping $\pi : S^D \to S^D = M$.

Notice, that this is a particular example where Gurevich theorem (8.20) is applicable and there is a correspondence between homotopies (degree of mapping) to cohomologies (integral of D-form of volume).

The integer value Q is a topological invariant of the mapping $\pi(x)$ and can be used to write down the topological theta-term as

$$S_\theta[\pi] = i\theta Q[\pi], \tag{8.26}$$

where θ is a coupling constant.

8.6.3.2 WZW terms

Assume now that the spacetime is S^D but the target space is $M = S^{D+1}$. The field configuration is given by the mapping $\pi : S^D \to S^{D+1} = M$. We extend this mapping continuously to $\tilde{\pi} : D^{D+1} \to S^{D+1} = M$, where the physical spacetime is the boundary of the disk $S^D = \partial D^{D+1}$. We write

$$W_D[\pi] = \int_{D^{D+1}} \tilde{\pi}^* \omega^{D+1}, \tag{8.27}$$

where $\tilde{\pi}^*\omega^{D+1}$ is a pullback of the volume form on the target space to the auxiliary disk D^{D+1}. The expression (8.27) is multiply-valued. Indeed, suppose that we have another extension of $\pi(x)$ to $\tilde{\pi}'$ on $D^{D+1\prime}$. We have

$$\begin{aligned} \int_{D^{D+1}} \tilde{\pi}^* \omega^{D+1} \quad - \quad & \int_{D^{D+1\prime}} \tilde{\pi}^{*\prime} \omega^{D+1} = \int_{S^{D+1}} \tilde{\pi}^* \omega^{D+1} \\ = \quad & \int_{M=S^{D+1}} \omega^D = integer. \end{aligned} \tag{8.28}$$

We see that $W_D[\pi]$ is defined modulo integer and the action

$$S_{WZW}[\pi] = i2\pi k W_D[\pi] \tag{8.29}$$

is a well defined object provided the coupling constant k is integer. This is WZW term. To generalize this construction to arbitrary target space M one needs the existence of closed but not exact form ω^{D+1} on M, i.e., $H^{D+1}(M, R) \neq 0$.

8.7 Exercises

Exercise 1: Forms on R^3

Consider the oriented euclidian space R^3 (with a given scalar product). Every vector $\vec{A} \in R^3$ determines a 1-form by $\omega^1_{\vec{A}}(\vec{\xi}) = (\vec{A} \cdot \vec{\xi})$ and a 2-form by $\omega^2_{\vec{A}}(\vec{\xi}_1, \vec{\xi}_2) = (\vec{A} \cdot [\vec{\xi}_1 \times \vec{\xi}_2])$.

Show that

$$a) \qquad \omega^1_{\vec{A}} \wedge \omega^1_{\vec{B}} = \omega^2_{[\vec{A} \times \vec{B}]},$$

$$b) \qquad \omega^1_{\vec{A}} \wedge \omega^2_{\vec{B}} = (\vec{A} \cdot \vec{B}) x_1 \wedge x_2 \wedge x_3.$$

Exercise 2: Vector calculus on 3-dimensional manifold

In a 3-dimensional oriented riemannian space M, every vector field $\vec{A}(x)$ corresponds to a 1-form $\omega^1_{\vec{A}}$ and $\omega^2_{\vec{A}}$ defined similarly to exterior forms in a previous exercise.

a) Show that

$$df = \omega^1_{\nabla f}, \qquad d\omega^1_{\vec{A}} = \omega^2_{\nabla \times \vec{A}}, \qquad d\omega^2_{\vec{A}} = (\nabla \cdot \vec{A}) \omega^3, \tag{8.30}$$

where f is a 0-form (scalar function) and ω^3 is a volume 3-form on M.

b) Identify the Stoke's formula applied to each equation in (8.30) with known formulae of vector calculus.

Exercise 3: Vector calculus on 3-dimensional manifold

Using differential forms and the results of previous exercises show that

$$\begin{aligned} a) \qquad \nabla \cdot [\vec{A} \times \vec{B}] &= (\nabla \times \vec{A}) \cdot \vec{B} - (\nabla \times \vec{B}) \cdot \vec{A}, \\ b) \qquad \nabla \times (a\vec{A}) &= (\nabla a) \times \vec{A} + a(\nabla \times \vec{A}), \\ c) \qquad \nabla \cdot (a\vec{A}) &= (\nabla a) \cdot \vec{A} + a(\nabla \cdot \vec{A}). \end{aligned} \tag{8.31}$$

Hint: Use the formula for the derivative of the product of forms.

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