Homework 12

Reading

JJS 3.1-3.3, 3.5-3.6.

Problem 1

Suppose that $g(t)$ is some matrix which smoothly depends on parameter t (time). Prove:

a) $\partial_t(g^{-1}) = -g^{-1}(\partial_t g)g^{-1}$.

b) If $g(t)$ is an orthogonal matrix, then the matrix of "angular velocity" $\Omega = g^{-1}\partial_t g$ is skew-symmetric, i.e., $\Omega^T = -\Omega$.

c) If $g(t)$ is a unitary matrix, then the matrix $\Omega = g^{-1}\partial_t g$ is skew-Hermitian, i.e., $\Omega^\dagger = -\Omega$.

Problem 2

Calculate using properties of Pauli matrices:

a) $\mathrm{tr}\Big((\boldsymbol{a}\cdot\boldsymbol{\sigma})(\boldsymbol{b}\cdot\boldsymbol{\sigma})(\boldsymbol{c}\cdot\boldsymbol{\sigma})\Big)$, where $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are some constant vectors.

b) $\exp\begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}$.

c) express an arbitrary 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as a linear combination of Pauli matrices (and a unit matrix).

Problem 3

For the state represented by the wave function

$$\psi = Ne^{-\alpha r^2}(x+y)z$$

a) Determine the normalization constant N as a function of parameter $\alpha > 0$.
b) Calculate the expectation values of $\boldsymbol{L}$ and $(\boldsymbol{L})^2$.
c) Calculate the variances of these quantities.

Problem 4

Explicitly work out the $\boldsymbol{J}$ matrices for $j = 1/2$, $j = 1$, and $j = 3/2$.

Problem 5

Show that in a state with some definite value of L_z the mean values of $L_{x,y}$ are zero.
Hint: use commutation relations.

Problem 6

a) Write down the eigenstates of S_x in terms of the basis $|1/2, \pm 1/2\rangle$ of the eigenstates of S_z for spin 1/2 system.

b) Write down the eigenstates of L_x in terms of the basis $|1, m\rangle$ $(m = 0, \pm 1)$ of the eigenstates of L_z for a system with angular momentum 1.

Problem 7

Let us define spherical functions in momentum representation as $\tilde{Y}_l^m(\tilde{\theta}, \tilde{\phi}) = \langle \hat{\tilde{n}} | l, m\rangle$, where $\hat{\tilde{n}}$ is the directional vector along the momentum of a particle. Find an explicit form of spherical functions in momentum representation.