Homework 15

Problems with stars are not for credit and will NOT be graded.

Reading
JJS 3.4, 4.1-4.2.

Problem 1
a) Find the reduced density matrix for the first spin 1/2 in the pure state of two spin 1/2 given by the (unnormalized) state vector
\[ |+\rangle \otimes |\rangle + 2 |\rangle \otimes |+\rangle + i |+\rangle \otimes |\rangle \].
Write it down as a 2 × 2 matrix.

b) What are the expectation values of components of \( S \) for the first spin in this state?

c) Find the entropy (entanglement) corresponding to the found reduced density matrix.

Hint: in c) find the eigenvalues of the density matrix.

Problem 2
a) Consider a mixed ensemble of spin 1/2 systems. Suppose that the ensemble averages \([S_x]\), \([S_y]\), and \([S_z]\) are all known. Find the density matrix of this ensemble.

b) What is the condition on \([S_j]\) so that the ensemble is pure?

c) Find the entropy of this ensemble.

Problem 3
Consider an ensemble of spin 1 systems. The density matrix is now 3 × 3 matrix. How many independent (real) parameters are needed to characterize the density matrix? What must we know in addition to \([S_x]\), \([S_y]\), and \([S_z]\) to characterize this ensemble completely?

Problem 4
Prove that if the dynamics of an ensemble is governed by the Schrödinger equation an ensemble which is pure at \( t = 0 \) cannot evolve into a mixed ensemble.

Problem 5
A quantum mechanical state \( \Psi \) is known to be a simultaneous eigenstate of two Hermitian operators \( A \) and \( B \) which anticommute,
\[ AB + BA = 0. \]
What can you say about the eigenvalues of $A$ and $B$ for state $\Psi$? Illustrate your point using the parity operator (which can be chosen to satisfy $\pi = \pi^{-1} = \pi^\dagger$) and the momentum operator.

*Problem 6*

Consider a symmetric rectangular double-well potential:

$$V = \begin{cases} +\infty & \text{for } |x| > a + b; \\ 0 & \text{for } a < |x| < a + b; \\ V_0 > 0 & \text{for } |x| < a. \end{cases}$$

Assuming that $V_0$ is very high compared to the quantized energies of low-lying states, obtain an approximate expression for the energy splitting between two lowest-lying states.

**Problem 7**

State the selection rules for the matrix element

$$\langle \alpha'; j', m'| z |\alpha; j, m \rangle,$$

where $z$ is the component of position operator, $j, m$ are eigenvalues of $J^2$ and $J_z$ respectively and $\alpha$ are all other quantum numbers.