Homework 16

Problems with stars are not for credit and will NOT be graded.

Reading

JJS 4.3-4.4.

Problem 1

The tight binding Hamiltonian on the 1d lattice with periodic boundary conditions is given by

$$H = -W \sum_{n=1}^{N} \Big(e^{i\theta} |n\rangle \langle n+1| + e^{-i\theta} |n+1\rangle \langle n| \Big),$$

where W and θ are some real parameters and $|n\rangle$ form an orthonormal basis of the Hilbert space "state on site n" and there is an identification $|n\rangle \equiv |n+N\rangle$. This Hamiltonian is obviously translationally invariant (one can change $n \to n+1$). Use this translational invariance to find the spectrum of the system. Namely,

a) Write down an operator that translates the system by one unit to the right and check explicitly that it commutes with the Hamiltonian.

b) Find the spectrum of the translation operator (use $T^N = 1$).

c) Use previous results to find the spectrum of the Hamiltonian.

Problem 2

Consider the tight-binding model on three sites

$$H = -W \sum_{n=1}^{3} \Big(e^{i\theta} |n\rangle \langle n+1| + e^{-i\theta} |n+1\rangle \langle n| \Big).$$

Assume that the density matrix of a system at $t = 0$ is given by

$$\rho(t=0) = \frac{1}{3} \left(2\, |1\rangle \otimes \langle 1| + |2\rangle \otimes \langle 2| \right).$$

Find the density matrix as a function of time $\rho(t)$.

Problem 3

Assume that some Hamiltonian is symmetric with respect to some discrete group of transformations. The group is generated by two elements S and T such that there are relations $S^2 = 1$, $T^3 = 1$ and $TST = S$. For example, S, T, T^2, ST are all different elements of the group.

a) What is the spectrum of operator S? of operator T?

b) Show that, generally, one expects to find doubly degenerate states in the spectrum of Hamiltonian of the system. What are eigenvalues of T for those degenerate states? (Hint: consider $S|\alpha\rangle$ with $|\alpha\rangle$ being an eigenstate of H.)

c) Can you give an example of the system having this symmetry? In your example, what is the meaning of S and T symmetry operations?

Problem 4

Let $\phi(\boldsymbol{p}')$ be the momentum-space wave function for state $|\alpha\rangle$, that is, $\phi(\boldsymbol{p}') = \langle \boldsymbol{p}'|\alpha\rangle$. Is the momentum-space wave function for the time reversed state $\Theta |\alpha\rangle$ given by $\phi(\boldsymbol{p}')$, $\phi(-\boldsymbol{p}')$, $\phi^*(\boldsymbol{p}')$, or $\phi^*(-\boldsymbol{p}')$? Justify your answer.

Problem 5

The Hamiltonian for a spin 1 system is given by

$$H = AS_z^2 + B(S_x^2 - S_y^2).$$

Solve this problem *exactly* to find the normalized energy eigenstates and eigenvalues. (A spin-dependent Hamiltonian of this kind actually appears in crystal physics.) Is this Hamiltonian invariant under time reversal? How do the normalized eigenstates you obtained transform under time reversal?