Homework 13

Reading

JJS 3.1-3.3, 3.5-3.7, 3.4.

Problem 1

A Hamiltonian of some system is given by

$$H = \frac{\boldsymbol{L}^2}{2I} + f\boldsymbol{L}\cdot\boldsymbol{S},$$

where we assume that I and f are some constants, $\boldsymbol{L}$ is an orbital angular momentum of the system and $\boldsymbol{S}$ is its spin operator.

a) Find the spectrum and degeneracy of eigenstates for $f = 0$.

b) Find the spectrum and degeneracy of eigenstates for finite f.

c) Let us consider the state with given l ($\boldsymbol{L}^2 = \hbar^2 l(l+1)$) and s ($\boldsymbol{S}^2 = \hbar^2 s(s+1)$) and $f = 0$. How will the energy level split if $f \neq 0$.

Problem 2

Two p-electrons are in the coupled angular momentum state $|l_1, l_2; l, m\rangle = |1, 1; 1, -1\rangle$. If measurement is made of L_{1z} in this state, what values may be found and with what probability will these values occur?

Problem 3

Consider the function $Y(\theta,\phi) = Y_2^2(\theta,\phi)Y_1^{-1}(\theta,\phi)$. Write it down explicitly in terms of elementary functions of θ and ϕ. Expand the result as a linear combination of $Y_l^m(\theta,\phi)$.

Problem 4

Consider the operators a, $a^\dagger$, b, and $b^\dagger$ obeying the following commutation relations $[a, a^\dagger] = [b, b^\dagger] = 1$ with other commutators giving zero. We define the operators

$$\begin{aligned} J_+ &\equiv \hbar\, a^\dagger b, \\ J_- &\equiv \hbar\, b^\dagger a, \\ J_z &\equiv \frac{\hbar}{2}(a^\dagger a - b^\dagger b). \end{aligned}$$

a) Show that the commutation relation between J-operators are the ones for the components of angular momentum.

b) Express the operator $\boldsymbol{J}^2 \equiv J_z^2 + \frac{1}{2}(J_+J_- + J_-J_+)$ in terms of the operator $N \equiv a^\dagger a + b^\dagger b$.

Problem 5

a) Find the reduced density matrix for the first spin 1/2 in the pure state of two spin 1/2 given by the (unnormalized) state vector

$$|+\rangle \otimes |-\rangle + 2\,|-\rangle \otimes |+\rangle + i\,|+\rangle \otimes |+\rangle\,.$$

Write it down as a 2×2 matrix.

b) What are the expectation values of components of $\boldsymbol{S}$ for the first spin in this state?

c) Find the entropy (entanglement) corresponding to the found reduced density matrix.

Hint: in c) find the eigenvalues of the density matrix.

Problem 6

a) Consider a mixed ensemble of spin 1/2 systems. Suppose that the ensemble averages $[S_x]$, $[S_y]$, and $[S_z]$ are all known. Find the density matrix of this ensemble.

b) What is the condition on $[S_j]$ so that the ensemble is pure?

c) Find the entropy of this ensemble.