Physics 556: Solid State Physics II

Homework 1

Read: FW 1,2

"**FW**" refers to Fetter, Walecka book.
Problems with stars are not for credit and will NOT be graded.

Exercise 1: Bogoliubov transformation

Suppose a is a canonical bose operator. Find the condition on the real coefficients u and v of the transformation

$$\begin{aligned} b &= ua + va^\dagger, \\ b^\dagger &= ua^\dagger + va. \end{aligned} \tag{1}$$

so that the transformation is canonical (i.e., b is also a canonical bose operator).

Exercise 2: Hidden oscillator

Using (1) diagonalize the Hamiltonian

$$H = \omega\left(a^\dagger a + \frac{1}{2}\right) + \frac{1}{2}\Delta(a^\dagger a^\dagger + aa), \tag{2}$$

by transforming it into the form $H = \tilde{\omega}\left(b^\dagger b + \frac{1}{2}\right)$. Find $\tilde{\omega}$, u and v in terms of ω and Δ. What happens when $\omega = \Delta$?

*Exercise 3: Canonical transformation of fermions

Let us now consider the transformation, where $a_{1,2}$ are canonical fermi operators

$$\begin{aligned} b_1 &= ua_1 + va_2^\dagger, \qquad & b_2 &= ua_2^\dagger - va_1, \\ b_1^\dagger &= ua_1^\dagger + va_2, \qquad & b_2^\dagger &= ua_2 - va_1^\dagger. \end{aligned} \tag{3}$$

Find the condition on the real coefficients u and v of the transformation so that the transformation is canonical (i.e., $b_{1,2}$ are also canonical fermi operators).

Exercise 4: Holstein-Primakoff transformation

Suppose that a is the canonical bose operator. Find the commutation relations $[S^+, S^z]$, $[S^-, S^z]$, and $[S^+, S^-]$ of the operators S^+, S^-, and S^z defined by the following relations.

$$S^+ = \hbar\sqrt{2S}\sqrt{1 - \frac{a^\dagger a}{2S}}\, a, \tag{4}$$

$$S^- = \hbar\sqrt{2S}a^\dagger\sqrt{1 - \frac{a^\dagger a}{2S}}, \tag{5}$$

$$S^z = \hbar(S - a^\dagger a), \tag{6}$$

where S is some real number. What is the spectrum (all eigenvalues) of the operator $(S^z)^2 + \frac{1}{2}S^+S^- + \frac{1}{2}S^-S^+$?

Exercise 5: Phonons

Consider the chain of atoms of the mass m connected by identical springs of the stiffness K.

$$H = \sum_{j=-\infty}^{+\infty} \left\{ \frac{\hat{p}_j^2}{2m} + \frac{K}{2}(\hat{x}_j - \hat{x}_{j+1})^2 \right\}, \tag{7}$$

where $\hat{p}_j = -i\hbar \frac{\partial}{\partial x_j}$ are quantum momenta operators.

Rewrite the Hamiltonian of the chain in terms of bosonic creation and annihilation operators

$$\hat{x}_j = \sqrt{\frac{\hbar}{m\omega}} \frac{a_j + a_j^\dagger}{\sqrt{2}}, \qquad \hat{p}_j = \sqrt{\hbar m\omega} \frac{a_j - a_j^\dagger}{i\sqrt{2}}. \tag{8}$$

Find the canonic transformation of bosonic operators diagonalizing the Hamiltonian. Find the spectrum of phonons and determine the energy of the ground state of the chain.

Hint: Use Fourier transform.

Exercise 6: Fock space of fermions.

Let us assume that c_j, $c_j^\dagger$ are fermionic operators and $j = 1, 2, 3, \ldots$ labels single particle states. We use the convention

$$|111110000\ldots\rangle = c_1^\dagger c_2^\dagger c_3^\dagger c_4^\dagger c_5^\dagger |vacuum\rangle.$$

a) Evaluate $c_3^\dagger c_6 c_4 c_6^\dagger c_3 |111110000\ldots\rangle$.
b) Write $|1101100100\ldots\rangle$ in terms of excitations about the "filled Fermi sea" $|111110000\ldots\rangle$. Interpret your answer in terms of electron and hole excitations.
c) Find $\langle\psi|\hat{N}|\psi\rangle$ where $|\psi\rangle = A|100000\ldots\rangle + B|111000\ldots\rangle$ and $\hat{N} = \sum_j c_j^\dagger c_j$.

Exercise 7: 2d electron gas.

Given the particle density $n = N/Area$ of a two-dimensional ideal Fermi gas (with spin $s = 1/2$) find its Fermi wavevector $k_F = p_F/\hbar$ and Fermi energy ϵ_F. What is the density of states $\nu(\epsilon_F)$ of such a gas at Fermi energy?

The Quantum Hall effect is observed at low temperatures in a quasi-two-dimensional electron gas, usually created in doped semiconductors. A typical two-dimensional electron density in such system is $2 \times 10^{11}\text{cm}^{-2}$. Calculate numerically for this "free electron" gas k_F, ϵ_F (in eV), and degeneracy temperature T_F (in K). Assume that the effective electron band mass is just a free electron mass.

Exercise 8: Spin in rotating field I.

A particle with spin $s = 1/2$ and magnetic moment μ is placed in the constant vertical and rotating horizontal magnetic fields[1] $B = (B_1 \cos\omega t, B_1 \sin\omega t, B_0)$.

$$H = \mu\vec{\sigma} \cdot \vec{B}(t). \tag{9}$$

Write down the Schroedinger equation in the interaction representation considering the alternating part of the magnetic field as "perturbation".

*Exercise 9: Spin in rotating field II.

Solve the Shroedinger equation obtained in the previous problem. Suppose that at $t = 0$ the particle is in the state with spin "up". What is the probability of finding it in the "down" state at the time $t > 0$? Consider separately the case of resonance $\omega = 2\mu B_0$.

[1] This situation occurs, e.g., in NMR experiments.