

1. In class, we did a “microscopic” mean field theory, starting with the Heisenberg Hamiltonian $H = -J \sum \vec{\sigma}_i \cdot \vec{\sigma}_j$, where the sum is over all nearest neighbor pairs (each pair counted once). The result is a mean field formula for the temperature dependence of the “order parameter” $\langle M \rangle$, or its dimensionless form $m = M/M_0$, where $M_0 = n\mu_B$ and n is the density of spins. The formula is $m = \tanh(m/t)$, where $t = T/T_C$, and $T_C = zJ/k_B$, with z the number of first neighbors.

(a) By solving $m = \tanh(m/t)$ graphically for $m(t)$ at a few values of t , make a graph that shows all the roots for $t > 0$.

(b) Using the fact that Fe (iron) has the bcc structure and has Curie temperature $T_C = 1043\text{K}$, what value of exchange constant J is expected in the mean field theory? Do the same for CrO_2 (rutile structure, $T_C = 386\text{K}$).

2. Kittel problem 8, p. 320 **Paramagnetism of S=1 system.** From the clue “paramagnetism” you should deduce that this problem is about non-interacting spins. These spins occur on objects whose dynamics is irrelevant to the magnetization – such as oxygen atoms in vapor.

(a) Find the magnetization as a function of magnetic field H and temperature T for a system of N spins in volume V , with $S = 1$ (quantized with $S_z = 1, 0, -1$). Their moment is μ , and concentration $n = N/V$.

(b) In the limit $\mu H \ll k_B T$, obtain the constant C of the Curie law $\chi = C/T$.

3. **The magnetic instability of the interacting electron gas.** “Interacting” means that electrons feel each other via the repulsive $1/r$ Coulomb potential. Overall charge neutrality is achieved by a rigid compensating background charge. As discussed in class, the Hartree-Fock equations have a self-consistent solution when the occupied Hartree-Fock orbitals are plane waves $\psi = \exp(i\vec{k} \cdot \vec{r})/\sqrt{V}$, V being the volume of the sample. It was not done in class, but this solution still works if the number of occupied spin-up states differs from the number of occupied spin-down states. The self-consistent computation of the energies is done in a parallel way. This permits a net ferromagnetic spin density $m = n_\uparrow - n_\downarrow$, where the density of electrons $n = N/V$ is separated into unequal up and down densities, with $n = n_\uparrow + n_\downarrow$. It then needs to be examined whether the polarized solution or the unpolarized solution has lower energy. The total energy equals $E_{tot} = E_K + E_X$, where the separate kinetic and exchange terms are

$$E_K = \sum_{\vec{k}}^{\text{spins } \uparrow \text{ occupied}} \frac{\hbar^2 k^2}{2m} + \sum_{\vec{k}}^{\text{spins } \downarrow \text{ occupied}} \frac{\hbar^2 k^2}{2m}$$

$$E_X = -\frac{e^2}{2} \sum_{\vec{k}, \vec{k}'}^{\text{both spins } \uparrow \text{ occupied}} \frac{1}{|\vec{k} - \vec{k}'|} - \frac{e^2}{2} \sum_{\vec{k}, \vec{k}'}^{\text{both spins } \downarrow \text{ occupied}} \frac{1}{|\vec{k} - \vec{k}'|}$$

For the unpolarized gas, we did the integrals in class, finding the result

$E_{tot}/N = (3/5)\epsilon_F - (3/4\pi)e^2 k_F$. For the polarized gas, this can be generalized to

$$\frac{E_{tot}}{N} = \frac{3}{5} \left(\frac{n_{\uparrow}}{n} \varepsilon_{F\uparrow} + \frac{n_{\downarrow}}{n} \varepsilon_{F\downarrow} \right) - \frac{3e^2}{4\pi} \left(\frac{n_{\uparrow}}{n} k_{F\uparrow} + \frac{n_{\downarrow}}{n} k_{F\downarrow} \right).$$

where $k_{F\uparrow} = (6\pi^2 n_{\uparrow})^{1/3}$, and the other notation follows logically.

(a) Express the total energy in terms of the electron density n and the spin density m . Evaluate your answer in the limiting cases $m=0$ and $m=n$.

(b) Find the density at which the total energy of the fully polarized gas ($m=n$) equals that of the unpolarized gas ($m=0$). For what densities is the polarized electron gas stable?