Physics 501 final exam Answers Monday December 17, 2012, 8:00 – 10:45 am

1. 2-d Kinematics; rigid body. A stick has mass *M* and moment of inertia *I*. It is oriented along the *y*-direction. It is free in the *x-y* plane, but has zero kinetic energy for times t < 0. The center of mass of the stick is at *(x,y)=(0,0)* for *t < 0*.

a. At time *t = 0* it experiences an impulsive force $F_x = P_0\,\delta(t)$, $F_y = 0$. This force is localized at a point on the stick with *y*-coordinate equal to y_0. What is the energy of the stick at *t > 0*?

A. The impulse $\int dt\vec{F} = P_0\hat{x}$ causes a change in *x*-momentum from *0* to P_0, or a change in linear kinetic energy of $P_0^2/2M$. There is also an angular impulse $\int dt\vec{r}\times\vec{F} = y_0P_0\hat{z}$. This causes *z*-angular momentum to increase from *0* to y_0P_0. So there is angular kinetic energy also, and the total $KE = P_0^2/2M + y_0^2P_0^2/2I$.

b. Suppose the source of the impulse was a collision [this should have said, "an elastic collision"] with a particle of mass *m*, which was moving in the *x-y* plane, in the *x* direction. In terms of $P_0, y_0, m, M,$ and *I*, what was the initial velocity v_0 of this particle?

B. Linear momentum is conserved, $\vec{p}_0 = \vec{p}_f + P_0\hat{x}$. Since the incident particle's momentum $\vec{p}_0$ was in the x-direction, its final momentum $\vec{p}_f$ is also in the *x*-direction. So momentum conservation gives us one equation for two unknowns. z-angular momentum is also conserved, $y_0p_0 = y_0p_f + y_0P_0$. This does not give any additional information. We have to use energy conservation, $p_0^2/2m = p_f^2/2m + P_0^2/2M + y_0^2P_0^2/2I$. Combining energy and momentum conservation equations, we find $v_0 = p_0/m = (\frac{1}{2M}+\frac{1}{2m})P_0 + y_0^2P_0/2I$.

2. Dynamics; rigid body. Suppose a rigid body has three unequal principle moments of inertia, $I_1 < I_2 < I_3$. Show that free rotations (no forces) around an axis close to the I_1 or I_3 axes are simple, but around an axis close to I_2, the body evolves in a more complicated way.

A. This was done in Landau and Lifshitz, and in class. The kinetic energy in the inertial frame of the center of mass is $E = L_1^2/2I_1 + L_2^2/2I_2 + L_3^2/2I_3$. The (vector) angular momentum is also conserved, but the components (L_1,L_2,L_3) are not separately conserved, since they are changing in time as the rigid body rotates. However, the magnitude, which is conserved, is a function of the components, $L^2 = L_1^2 + L_2^2 + L_3^2$. In (L_1,L_2,L_3)-space, L^2 is a sphere and $2I_1E$ is an ellipsoid. The conservation laws constrain the motion to the intersection of the sphere and ellipsoid. If at some time, $L_2^2 << L_1^2$ and $L_3^2 << L_1^2$, then the sphere L^2 is barely intersecting the ellipsoid in the directions of its shortest axis, and motion is confined to this region. The vector (L_1,L_2,L_3) is approximately $(L_1,0,0)$. This is stable rotation around the axis of least inertia. If at some time, $L_1^2 << L_3^2$ and $L_2^2 << L_3^2$, then the sphere L^2 is barely intersecting the ellipsoid in the directions of its longest axis, and motion is confined to this region. The vector (L_1,L_2,L_3) is approximately $(0,0,L_3)$. This is stable rotation around the axis of greatest inertia. In the intermediate case, the intersection is not confined to the vicinity of the intermediate axis, so the motion is not a simple rotation with (L_1,L_2,L_3) approximately $(0,L_2,0)$. This is unstable rotation around the axis of intermediate inertia.

Alternate proof: Many students wrote the Euler equations of motion. Only one successfully used these to prove the required behavior.

The Euler equation of motion for rigid rotation is

$$d\vec{L}/dt + \vec{\Omega} \times \vec{L} = \vec{\tau}. \tag{1}$$

This is for the angular momentum in a frame rotating (by $\vec{\Omega}$) relative to the lab frame. For free rotation, the torque $\vec{\tau}$ is zero, and we have, in the frame rotating with the rigid body, using as axes the principle axes:

$$\frac{d\Omega_1}{dt} = \frac{I_2 - I_3}{I_1}\Omega_2\Omega_3$$
$$\frac{d\Omega_2}{dt} = \frac{I_3 - I_1}{I_2}\Omega_3\Omega_1$$
$$\frac{d\Omega_3}{dt} = \frac{I_1 - I_2}{I_3}\Omega_1\Omega_2 \tag{2}$$

These equations correctly describe the motion, and are consistent with energy conservation and conservation of L^2 used above. To use them directly to prove stability or instability of rotations requires care. What you need to do is assume that initially $\vec{\Omega}$ is near one of the axes, say axis 1, and show that the other components (Ω_2 and Ω_3, which by assumption are initially small compared to Ω_1) do not increase, but rather oscillate, therefore staying small. The Euler equations then say that (at least initially) $d\Omega_1/dt$ is small, so Ω_1 is constant, and

$$
\begin{aligned}
d\Omega_2/dt &= C_2\Omega_3 \\
d\Omega_3/dt &= C_3\Omega_2 \\
d^2\Omega_2/dt^2 &= -\omega_1^2\Omega_2 \\
d^2\Omega_3/dt^2 &= -\omega_1^2\Omega_3
\end{aligned}
\tag{3}
$$

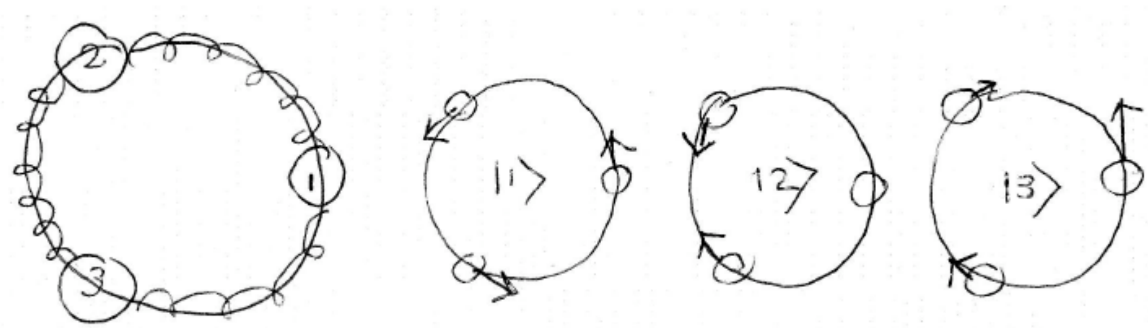
Where $C_2 = \Omega_1(I_3 - I_1)/I_2$ and $C_3 = \Omega_1(I_1 - I_2)/I_2$ are approximately constants. Notice that because we choose $I_1 < I_2 < I_3$, the product $\omega_1^2 = -C_2C_3$ is positive. Then the equations 3 show that the frequencies Ω_2 and Ω_3 evolve by oscillating at frequency ω_1, so the angular velocity $\vec{\Omega}$ lies stably close to the 1 axis. The argument can be repeated for the 2 and 3 axes. The product $\omega_3^2 = -C_1'C_2'$ is also positive, but the product $\omega_2^2 = -C_3''C_1''$ is negative. When oscillations start near the 2 axis, the small components Ω_1 and Ω_3 do not oscillate, but grow exponentially, indicating instability. The motion may appear chaotic. However, it is not actually chaotic. The equations are "integrable", and I suspect therefore that nearby initial conditions do not diverge exponentially.

3. Normal modes of vibration. Three equal masses *m* are interconnected on a circle by identical massless springs of force constant *k*. The unstretched springs have length *a*. The circle has circumference *3a*. The masses are constrained to move in a circle, and the springs similarly constrained, bending to conform to the circle. The problem is to determine the normal mode frequencies, and to draw pictures to illustrate the corresponding eigenvectors.

a. One solution of Newton's laws is "trivial." Explain what it is. Explain what is the corresponding eigenvector.

b. Find other solutions by any method you like. For example, you can guess. Or you can exploit the fact that the remaining eigenvectors are orthogonal to the trivial one, which enables you to write Newton's laws in the non-trivial subspace as a 2 x 2 matrix.

A. The system, and a particular choice of normal mode eigenvectors, is shown below.



The potential energy is

$$V = \frac{k}{2}\left[(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2\right].$$

This can be written in matrix form as $\langle x|\hat{V}|x\rangle$, with

$$\hat{V} = \frac{k}{2}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Eigenvectors of this matrix, not normalized, and in the same convention as the picture above, are

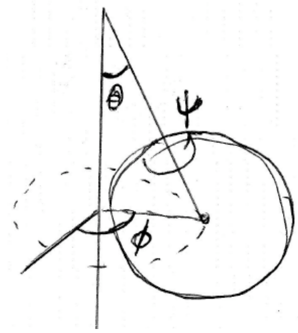
$$|1\rangle = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad |3\rangle = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

The corresponding eigenvalues are $\lambda_1 = \omega_1^2 = 0$, and $\lambda_2 = \lambda_3 = \omega_2^2 = \omega_3^2 = 3k$. The first of these is the "trivial" eigenvector, corresponding to a uniform rotation, with no restoring force, and therefore normal mode frequency equal to zero. The other two are degenerate with frequency $\omega = \sqrt{(}3k/m)$. There are many ways to find the last two eigenvectors and eigenvalues. The most elementary is, after finding the trivial vector, construct two orthonormal vectors orthogonal to the trivial vector, for example, $(x_1, x_2, x_3) = 1/\sqrt{(}2)(0, 1, -1)$ and $(x_1, x_2, x_3) = 1/\sqrt{(}6)(2, -1, -1)$. Then compute the matrix elements of $\hat{V}$ for these two vectors, giving a 2×2 matrix. Equivalently, but more sophisticated, just guess that the vector $|2\rangle$ shown above might be an eigenvector, and verify that it is. Rotational symmetry then says that two other similar vectors $(x_1, x_2, x_3) = (-1, 0, 1)$ and $(x_1, x_2, x_3) = (1, -1, 0)$ must also be eigenvectors with the same eigenvalue. Yes they are. But they are not orthogonal to each other or to the first non-trivial eigenvector. The three are overcomplete, and span a subspace of dimension 2. Any two orthogonal vectors in this subspace will do. One final way to do it, familiar to solid state physicists, is that Bloch's theorem tells us that $(x_1, x_2, x_3) = 1/\sqrt{(}3)(1, \exp(ika), \exp(2ika))$ should work, with $ka = \pm 2\pi/3$. This gives vectors orthogonal to the trivial vector, and are eigenvectors with the appropriate eigenvalue.

4. Spherical pendulum. This is a point mass that moves under gravity (g) in two dimensions. [Note: this could have been stated better: "in three dimensions, with two degrees of freedom."] Think of a massless stick of length R, attached at one end to the origin, but free to move to any angle. A point mass m is at the far end.

a. Choose appropriate coordinates and write the Lagrangian. Use this to write the equations of motion.
b. Find the conjugate momenta and derive the Hamiltonian from them. Use this to write the equations of motion.
c. Show how the problem can be reduced to one-dimensional motion in an effective potential, whose parameters depend on initial conditions. What is this effective potential, and what initial conditions need specification?
d. Now replace the point mass by a sphere of radius r (where $r < R/2$). Suppose the sphere can rotate around the axis defined by the stick. This axis passes through the

center of the sphere. (The moment of inertia of a sphere around its axis is $2mr^2/5$.) Repeat the exercise in part a.



A. Relevant coordinates are θ and ϕ. Later, when the rotating sphere replaces the point mass, the angle ψ will also be needed. The Lagrangian and the equations of motion are

$$\mathcal{L} = (mR^2/2)(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + mgR\cos\theta$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \frac{\partial\mathcal{L}}{\partial\theta} = mR^2\ddot{\theta} = -mgR\sin\theta + mR^2\sin\theta\cos\theta\dot{\phi}^2$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \frac{\partial\mathcal{L}}{\partial\phi} = \frac{d}{dt}mR^2\sin^2\theta\dot{\phi} = 0 = \frac{dL_\phi}{dt}.$$

B. Conjugate momenta and the Hamiltonian are

$$p_\theta = \frac{\partial\mathcal{L}}{\partial\dot{\theta}} = mR^2\dot{\theta} \qquad p_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = mR^2\sin^2\theta\dot{\phi}$$

$$\begin{aligned}\mathcal{H} &= \dot{\theta}p_\theta + \dot{\phi}p_\phi - \mathcal{L} \\ &= (mR^2/2)(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) - mgR\cos\theta \\ &= \frac{p_\theta^2}{2mR^2} + \frac{p_\phi^2}{2mR^2\sin^2\theta} - mg\cos\theta\end{aligned}$$

Hamilton's equations of motion are

$$\dot{\theta} = \frac{\partial\mathcal{H}}{\partial p_\theta} = \frac{p_\theta}{mR^2} \qquad \dot{\phi} = \frac{\partial\mathcal{H}}{\partial p_\phi} = \frac{p_\phi}{mR^2\sin^2\theta}$$

$$\dot{p}_\theta = -\frac{\partial\mathcal{H}}{\partial\theta} = -mgR\sin\theta + \frac{p_\phi^2\cos\theta}{mR^2\sin^3\theta} \qquad \dot{p}_\phi = -\frac{\partial\mathcal{H}}{\partial\phi} = 0$$

C. The angular momentum $L_\phi = p_\phi$ is a constant of the motion, so the variable ϕ can be eliminated. The equation for θ can then be written in two ways, the second one being the first integral of the first,

$$mR^2\ddot{\theta} = -\frac{\partial}{\partial\theta}U_{\text{eff}}(\theta) \qquad U_{\text{eff}}(\theta) = -mgR\cos\theta + \frac{p_\phi^2}{2mR^2\sin^2\theta}$$

$$E = mR^2\dot{\theta}^2/2 + U_{\text{eff}}(\theta)$$

The initial conditions $(\theta, \dot{\theta}, \phi, \dot{\phi})$ determine the numerical value of p_ϕ.

D. No student got this part completely right.

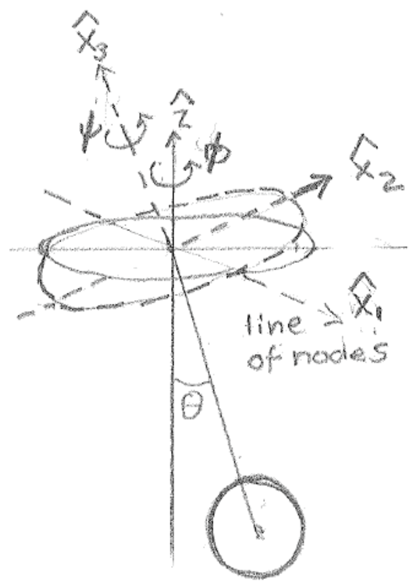


FIG. 1. Because of the symmetry of the pendulum, the axes $\hat{x}_1$ and $\hat{x}_2$ can be chosen not to rotate with the spin of the pendulum, but instead to rotate with the azimuthal angle ϕ. The axis $\hat{x}_1$ is chosen as the line of nodes, lying in the horizontal plane.

Consider a pendulum which can spin around its rigid support, which is a symmetry axis. The pendulum free to spin around its symmetry axis $\hat{x}_3$ is exactly equivalent to a symmetric top except gravity has been reversed. The inertia tensor has principal axes $\hat{x}_1$, $\hat{x}_2$, and $\hat{x}_3$, with moments I_1, I_1, and I_3. If the pendulum is a sphere, then $I_3 = 2mr^2/5$ where r is the sphere radius. The moments I_1 are $mR^2 + 2mr^2/5$, where R is the distance from the support to the center of mass. The kinetic energy is $(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2)/2$. The angular velocity is expressed with Euler angles which have vector directions not orthogonal to each other,

$$\vec{\Omega} = \dot{\theta}\hat{x}_1 + \dot{\phi}\hat{z} + \dot{\psi}\hat{x}_3 \tag{1}$$

The lab axis $\hat{z}$ lies in the $\hat{x}_2$ - $\hat{x}_3$ plane, with components $\sin\theta$ and $\cos\theta$ respectively, Thus the angular velocity is

$$\vec{\Omega} = \dot{\theta}\hat{x}_1 + \sin\theta\dot{\phi}\hat{x}_2 + (\cos\theta\dot{\phi} + \dot{\psi})\hat{x}_3, \tag{2}$$

and the kinetic energy is

$$\text{KE} = \frac{1}{2}\left[I_1\dot{\theta}^2 + I_1\sin^2\theta\dot{\phi}^2 + I_3(\cos\theta\dot{\phi} + \dot{\psi})^2\right] \tag{3}$$

We can now take the limit where the pendulum becomes a point mass. Then $I_3 \to 0$ and $I_1 \to mR^2$. We recover the familiar kinetic energy of the simple pendulum free to swing with two degrees of freedom described by θ and ϕ.

The rest of the problem is straightforward, once kinetic energy is correctly formulated.

5. Canonical transformation Here is an F_2 – type generating function (Φ – type in Landau-Lifshitz notation): $F_2(q,P) = \frac{1}{2}P^2 + \frac{1}{2}im\omega q^2 + \sqrt{2im\omega}\ qP$. Applied to a harmonic oscillator, it is quite a lot like the familiar raising/lowering operator algebra used in the quantum treatment. The Hamiltonian is $H = p^2/2m + m\omega^2 q^2/2$.

a. Solve for *q = q(P,Q)* and *p = p(Q,P).*
b. Find the new Hamiltonian and the equations of motion for *Q* and *P*.
c. Solve the equations of motion for *Q* and *P*, with initial conditions *q(0)=A* and *p(0)=0*.
d. Find *q(t)* and *p(t).*

A.

For an F_2 or $|\Phi$-type generating function, the relations are $p = \partial F_2/\partial q$ and $Q = \partial F_2/\partial P$. This gives

$$p = im\omega q + \sqrt{2im\omega}\, P \quad \text{and} \quad Q = \sqrt{2im\omega}\, q + P$$

Solving for q and p, we get

$$q = \sqrt{\frac{1}{2im\omega}}(Q - P) \quad \text{and} \quad p = \sqrt{\frac{im\omega}{2}}(Q + P) \quad \text{and} \quad \mathcal{H} = i\omega PQ$$

The equations of motion for the new canonical variables P and Q are

$$\dot{Q} = \partial\mathcal{H}/\partial P = i\omega Q \quad \text{and} \quad \dot{P} = -\partial\mathcal{H}/\partial Q = -i\omega P$$

The general solution is

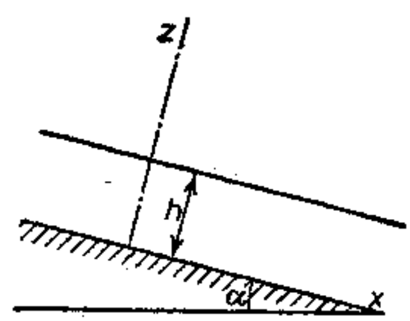
$$Q(t) = \mathcal{A}e^{i\omega t} \quad \text{and} \quad P(t) = \mathcal{B}e^{-i\omega t},$$

where $\mathcal{A}$ and $\mathcal{B}$ are complex constants. The given initial conditions are $q(0) = A$ and $p(0) = 0$. Solving for P and Q in terms of p and q,

$$P = \sqrt{\frac{1}{2im\omega}}\, p - \sqrt{\frac{im\omega}{2}}\, q \quad \text{and} \quad Q = \sqrt{\frac{1}{2im\omega}}\, p + \sqrt{\frac{im\omega}{2}}\, q$$

From these we find that $\mathcal{A} = -\sqrt{im\omega/2}\, A$ and $\mathcal{B} = \sqrt{im\omega/2}\, A$. These are then used to find the elementary results $q(t) = A\cos(\omega t)$ and $p(t) = -m\omega A\sin(\omega t)$.

6. Downhill flow A layer of fluid flows steadily downhill under gravity. The layer has thickness h, on top of a surface inclined by angle α. At the top surface, the pressure P_0 can be taken to be zero (the flow takes place in vacuum.) The boundary condition on the top surface is that the stress, $\eta dv_x/dz$, vanishes. The transverse dimension (y) is sufficiently long that we neglect any boundary effects in this direction and treat the problem as two dimensional.



a. What is the boundary condition on the lower surface ($z=0$, where the fluid touches the incline)?
b. Write the Navier-Stokes equation. Simplify it to a one-dimensional differential equation. Certain terms are zero. Explain why.
c. Find the solution that obeys the two boundary conditions.
d. Find the total discharge Q (in the x direction) per unit length (in the y-direction).
e. Compute the discharge Q (in liters/m s) for water (viscosity $\eta = 1.00 \times 10^{-3}$ Pa s, density $\rho = 1.00$ g/cm^3) if the height is 0.4cm and the angle is $\alpha = 1.7^\circ$ ($\sin 1.7^\circ = 0.03$). [1 liter = 10^{-3} m^3 = 10^3 cm^3]

A. On the lower surface, the velocity is zero, $v_x(z=0)=0$.

The Navier-Stokes equation (which assumes incompressibility) is

$$\rho\left[\frac{\partial\vec{v}}{\partial t}+\vec{v}\cdot\vec{\nabla}\vec{v}\right]=-\vec{\nabla}p+\rho\vec{g}+\eta\nabla^2\vec{v}$$

We use coordinates where the velocity vector is in the x direction. The flow is steady, so $\partial\vec{v}/\partial t=0$. The flow is translationally invariant in the x-direction, so $\nabla_x\vec{v}=0$. The velocity vector varies only in the z-direction, normal to the plane; $(\vec{v}\cdot\vec{\nabla})\vec{v}=\hat{x}v_x\nabla_x v_x=0$, so both terms on the left hand side are zero. The gravitational force $\rho\vec{g}=\rho g(-\cos\alpha\hat{z}+\sin\alpha\hat{x})$. The z-component of the Navier-Stokes equation just tells us that the pressure gradient is $\rho g\cos\alpha$, to compensate for the component of gravity perpendicular to the plane. There is no pressure gradient in the x-direction. The source of the pressure is confining walls, and there is no wall perpendicular to the x-direction to establish a gradient. The non-vanishing terms of the Navier-Stokes equation in the x-direction are $\rho g\sin\alpha+\eta\nabla_z^2 v_x=0$.
The general solution, and the solution obeying the boundary conditions, are

$$v_x=a+bz-(\rho g\sin\alpha/2\eta)z^2;\quad b-(\rho g\sin\alpha/\eta)h=0;\quad a=0;\quad v_x=(\rho g\sin\alpha/\eta)z(h-z/2).$$

To find the mass discharge $Q_\rho=\rho h\bar{v}_x$, or the volume discharge, $Q_V=h\bar{v}_x$, we need to average v_x over the z-direction.

$$\begin{aligned}Q_V &= \int_0^h v_x(z)=\rho g\sin\alpha h^3/3\eta\\ &= 10^3\text{kg/m}^3\times 9.8\text{m/s}^2\times 0.03\times(4\times 10^{-3}\text{m})^3/(3\times 10^{-3}\text{Pa s})=6\times 10^{-3}\text{m}^3/\text{m s}\\ &= 6\times 10^{-3}\text{m}^3/\text{m s}\times 10^3\text{liter/m}^3=6.\ \text{liter/m s}\end{aligned}$$