

50|-1 (25 points)

1a. (5 points)

$$S_k = \int_0^{t_1} \frac{1}{2} m (\omega_0 + v_1) - \frac{2v_1}{t_1} t \, t^2$$

$$= \frac{1}{2} m (\omega_0^2 + \frac{1}{3} v_1^2) t_1$$

$$\frac{\partial S_k}{\partial v_1} = 0 \Rightarrow \frac{1}{3} m v_1 t_1 = 0 \quad \dots \quad v_1 = 0$$

1b. (5 points)

$$S_U = \int_0^{t_1} mg \left[ \omega_0 + v_1 - \frac{2v_1}{t_1} t \right] \frac{t^2}{2} dt$$

$$= mg \left[ \frac{1}{2} (\omega_0 + v_1) t_1^2 - \frac{1}{3} v_1 t_1^2 \right]$$

$$= mg \left( \frac{1}{2} \omega_0 t_1^2 + \frac{1}{6} v_1 t_1^2 \right)$$

$$S = S_k - S_U = \frac{1}{2} m (\omega_0^2 + \frac{1}{3} v_1^2) t_1 - mg \left( \frac{1}{2} \omega_0 + \frac{1}{6} v_1 \right) t_1^2$$

$$\frac{\partial S}{\partial v_1} = 0 \Rightarrow \frac{1}{3} m v_1 t_1 - \frac{1}{6} m g t_1^2 = 0 \quad \dots \quad v_1 = \frac{1}{2} g t_1$$

1c. (5 points)

$$y = (\omega_0 + v_1) t - \frac{2v_1}{t_1} \frac{t^2}{2} + A \sin \frac{n\pi t}{t_1}$$

$$S = \int_0^{t_1} \left\{ \frac{1}{2} m \left[ (\omega_0 + v_1) - \frac{2v_1}{t_1} t + \frac{n\pi A}{t_1} \cos \frac{n\pi t}{t_1} \right]^2 - mg \left[ (\omega_0 + v_1) t - \frac{2v_1}{t_1} \frac{t^2}{2} + A \sin \frac{n\pi t}{t_1} \right] \right\} dt$$

$$= \int_0^{t_1} \left\{ \frac{1}{2} m \left[ (\omega_0 + v_1 - \frac{2v_1}{t_1} t)^2 + \frac{n^2 \pi^2 A^2}{t_1^2} \cos^2 \frac{n\pi t}{t_1} + 2(\omega_0 + v_1 - \frac{2v_1}{t_1} t) \frac{n\pi A}{t_1} \cos \frac{n\pi t}{t_1} \right] - S_U \right\} dt$$

$$= \frac{1}{2} m (\omega_0 + v_1)^2 t_1 + \frac{1}{2} m \frac{4v_1^2}{t_1^2} \cdot \frac{1}{3} t_1^3 - \frac{1}{2} m \cdot 2(\omega_0 + v_1) \frac{2v_1}{t_1} \frac{1}{2} t_1^2 + \frac{1}{2} \frac{n^2 \pi^2 A^2}{t_1^2} \cdot \frac{1}{2} m$$

$$- \frac{4v_1}{t_1} \cdot n\pi A \cdot \frac{t_1^2}{n^2 \pi^2} \cos \frac{n\pi t}{t_1} \Big|_0^{t_1} \cdot \frac{1}{2} m - S_U$$

$$S = \frac{1}{2} m v_0^2 t_1 + \frac{1}{2} m v_1^2 t_1 + \frac{m}{2\pi} n^2 \pi^2 A^2 - \frac{1}{2} m \frac{4v_1 A}{n\pi} [(-1)^n - 1] - mg(v_0 + v_1) \frac{1}{2} t_1^2 +$$

$$mg \frac{v_1}{n} \frac{1}{2} t_1^2 + mgA \frac{t_1}{n\pi} [(-1)^n - 1]$$

$$\frac{\partial S}{\partial v_1} = \frac{1}{2} m v_1 t_1 - \frac{2mA}{n\pi} [(-1)^n - 1] - mg \frac{1}{2} t_1^2 + \frac{1}{2} m g t_1^2 = 0$$

$$\frac{\partial S}{\partial A} = \frac{m}{2\pi} n^2 \pi^2 A - \frac{2m v_1}{n\pi} [(-1)^n - 1] + mg \frac{t_1}{n\pi} [(-1)^n - 1] = 0$$

$$n = \text{even} \Rightarrow \begin{cases} \frac{1}{2} m v_1 t_1 - \frac{1}{2} m g t_1^2 = 0 \\ \frac{m}{2\pi} n^2 \pi^2 A = 0 \end{cases} \Rightarrow \begin{cases} v_1 = \frac{1}{2} g t_1 \\ A = 0 \end{cases}$$

$$n = \text{odd} \Rightarrow \begin{cases} \frac{1}{2} m v_1 t_1 + \frac{4mA}{n\pi} - \frac{1}{2} m g t_1^2 = 0 \\ \frac{m}{2\pi} n^2 \pi^2 A + \frac{4m v_1}{n\pi} - \frac{2m g t_1}{n\pi} = 0 \end{cases} \Rightarrow \begin{cases} v_1 = \frac{1}{2} g t_1 \\ A = 0 \end{cases}$$

2a. (5 points)

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

$$m \ddot{x} + kx = 0 \Rightarrow x(t) = b \cos \omega_0 t \quad \begin{matrix} \Rightarrow \dot{x}(0) = 0! \\ (\omega_0 = \sqrt{\frac{k}{m}}, x(0) = b) \end{matrix}$$

$$\omega_0 t = \arccos\left(\frac{a}{b}\right)$$

2b. (5 points)

$$L = \frac{1}{2} m l^2 \dot{\phi}^2 + mgl \cos \phi$$

$$m l^2 \ddot{\phi} + mgl \sin \phi = 0 \quad x = l(\pi - \phi) \quad \sin \phi = \sin(\pi - \phi) \approx \pi - \phi$$

$$\ddot{\phi} + \frac{g}{l} \sin \phi = 0 \Rightarrow \ddot{x} - \frac{g}{l} x = 0 \Rightarrow \ddot{x} - \omega_0^2 x = 0 \quad (\omega_0 = \sqrt{\frac{g}{l}})$$

$$x = a \cosh(\omega_0 t) \quad \omega_0 t = \operatorname{arccosh}\left(\frac{b}{a}\right)$$

2C. (0 points)

$$\ddot{x} + \omega^2 x = 0, \quad \text{now } \omega \rightarrow \omega e^{i\theta}$$

$$\text{Set } x(0) = a, \quad \dot{x}(0) = 0 \Rightarrow x = a \cos \omega t$$

$$\text{at } x(t) = b, \quad \omega t = \arccos \frac{b}{a}$$

$$\text{for } \theta = 0, \quad \omega t = \arccos \frac{b}{a};$$

$$\text{for } \theta = \frac{\pi}{2}, \quad i\omega t = \arccos \frac{b}{a}, \quad \omega t = -i \arccos \frac{b}{a} = \text{arccosh} \frac{b}{a}$$

$$\arccos x = -i \ln(x + i\sqrt{1-x^2}) \iff \text{arccosh} x = \ln(x + \sqrt{x^2-1})$$