1. (10 points)

\[ \vec{\nabla} p = \rho \vec{g} \]  
\[ (1) \]

Ideal gas EOS: \( v = \frac{V}{m} = \frac{1}{\rho} \) specific volume, \( R^* = \frac{R}{M} \) specific gas constant

\[ pv = \frac{p}{\rho} = R^* T \]  
\[ (2) \]

\[ \frac{dp}{dz} \bigg|_T = R^* \frac{d\rho}{dz} = -\rho g \]  
\[ (3) \]

\[ R^* T \ln \frac{\rho(z)}{\rho_0} = -g(z - z_0) \]  
\[ (4) \]

\[ \rho(z) = \rho_0 \exp \frac{g}{R^* T} (z_0 - z) \]  
\[ (5) \]

For ideal N\(_2\) gas at \( z = 100m \) and \( T = 300K \), \( \rho(z)/\rho(0) = 0.989 \). 

Isothermal bulk modulus \( B_T = -v \left( \frac{\partial p}{\partial v} \right)_T = \rho \frac{\partial p}{\partial \rho} \bigg|_T \).

\[ \frac{dp}{dz} \bigg|_T = \frac{\partial p}{\partial \rho} \frac{d\rho}{dz} = \frac{B_T d\rho}{dz} = -\rho g \]  
\[ (6) \]

\[ \frac{1}{\rho(z)} - \frac{1}{\rho(0)} = \frac{g}{B_T} (z - z_0) \]  
\[ (7) \]

For water at \( z = -100m \), \( \rho(z)/\rho(0) = 1.00045 \). 

2. (5 points)

For the derivation, please refer to L.L., Fluid Mechanics, section 4.

\[ \frac{dT}{dz} < \frac{g}{c_p} \]  
\[ (8) \]

where \( c_p = \frac{7}{5} c_v \) and \( c_v = \frac{5}{2} R/M \). Critical size of the temperature gradient is thus \( -\frac{dT}{dz} < 0.0094K/m \).

Also, the derivation given in class is summarized below:

Let there be a temperature gradient \( dT/dz \) in an ideal gas under gravity. We know that hydrostatics requires \( dP/dz = -\rho g \) for this gas. We know that there will be a convective instability if a piece of gas, lifted adiabatically, finds its density greater than that of the gas it displaced. Or, the gradient of temperature is subject to convective instability if

\[ \frac{1}{V} \left( \frac{\partial V}{\partial z} \right)_{\text{adiabatic}} > \frac{1}{V} \left( \frac{\partial V}{\partial z} \right)_{\text{actual}} \]  
\[ (9) \]

where “actual” means the gradient of the system without any imposed displacement. This latter can be written

\[ \frac{1}{V} \left( \frac{\partial V}{\partial z} \right)_{\text{actual}} = \frac{1}{V} \left( \frac{\partial V}{\partial z} \right)_{T} + \frac{1}{V} \left( \frac{\partial V}{\partial T} \right)_{P} \left( \frac{\partial T}{\partial z} \right)_{\text{actual}} \]  
\[ (10) \]

For the adiabatic case, use \( pV^\gamma = \text{const} \) and the equation of hydrostatics to write

\[ \frac{1}{V} \left( \frac{\partial V}{\partial z} \right)_{\text{adiabatic}} = \frac{\rho g}{\gamma B}. \]  
\[ (11) \]

For the “actual” case, we need the ideal gas result

\[ \frac{1}{V} \left( \frac{\partial V}{\partial T} \right)_{P} = \frac{1}{T}. \]  
\[ (12) \]
This enables the condition for instability to be written
\[
\frac{\rho g}{\gamma P} > \frac{\rho g}{P} + \frac{1}{T} \left( \frac{\partial T}{\partial z} \right)_{\text{actual}}. \tag{13}
\]
Rearranging this equation, the condition for convective instability is
\[
-\left( \frac{\partial T}{\partial z} \right)_{\text{actual}} > \left( 1 - \frac{1}{\gamma} \right) \frac{\rho g T}{P}. \tag{14}
\]
The derivation above is the one given in class. Landau and Lifshitz solve the problem more generally in section 4 of *Fluid Mechanics*. Their result does not assume an ideal gas, and gives the condition for instability as
\[
-\left( \frac{\partial T}{\partial z} \right)_{\text{actual}} > \frac{\rho g T}{C_p V} \left( \frac{\partial V}{\partial T} \right)_{P}. \tag{15}
\]
If you work this out for an ideal gas, you recover the same result.

Numerically, the problem asks “Consider a large, thermally insulated tank of N\textsubscript{2} gas, under gravity, at \( P_0 = 1 \text{ atm} \) and \( T_0 = 300\text{K} \), with a uniform temperature gradient \( -dT/dz < 0 \). \ldots \) You can treat N\textsubscript{2} as a diatomic ideal gas, with \( C_p/C_V = \gamma = 7/5 \).” The numerical answer is that instability occurs when \( -dT/dz > 0.0094 \text{K/m} \).

3. (15 points)
Equation for the velocity potential is Laplace equation assuming irrotational flow:
\[
\nabla^2 \Phi = 0 \tag{16}
\]
Boundary conditions:
\[
\left. \frac{\partial \Phi}{\partial z} + \frac{1}{g} \frac{\partial^2 \Phi}{\partial t^2} \right|_{z=0} = 0 \tag{17a}
\]
\[
v_z|_{z=-h} = \left. \frac{\partial \Phi}{\partial z} \right|_{z=-h} = 0 \tag{17b}
\]
\[
\Phi = A \cos(kx - \omega t) \cosh k(z + h) \tag{18}
\]
\[
\omega^2 = gk \tanh kh \tag{19}
\]
It can be easily seen that in the limiting cases \( \lambda \ll h \) and \( \lambda \gg h \), \( \omega^2 = gk \) and \( \omega = \sqrt{ghk} \) respectively.
\[
\begin{align*}
v_y &= \frac{\partial \omega}{\partial k} = \frac{1}{2} \frac{g \tanh kh}{k} + \frac{gkh}{2\sqrt{gk \tanh kh \cosh^2 kh}} \tag{20a} \\
v_p &= \frac{\omega}{k} = \frac{\sqrt{2}}{k} \tanh kh \tag{20b}
\end{align*}
\]
Group velocity is shown below:

4. (15 points)
N-S equation reads:
\[
\vec{f} - \nabla p + \eta \nabla^2 \vec{v} = \rho \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right) \tag{21}
\]
Assume steady laminar flow along the axis of the cylinder \( \hat{y} \), then \( v_y = v, v_x = v_z = 0, \frac{\partial v_y}{\partial t} = 0 \). Ignore the effect of gravity, we thus have \( \vec{f} = 0 \). From the continuity equation, we have (for incompressible fluid) \( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \), and thus \( \frac{\partial v_y}{\partial y} = 0 \). N-S equations reduce to
\[
\begin{align*}
\frac{\partial p}{\partial y} - \eta \left( \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} \right) &= 0 \tag{22a} \\
\frac{1}{\rho} \frac{\partial p}{\partial x} - \frac{1}{\rho} \frac{\partial p}{\partial z} &= 0 \tag{22b}
\end{align*}
\]
From (22b), we know that $p = p(y)$, thus $\frac{\partial p}{\partial y} = \frac{dp}{dy}$, and we can further assume $\frac{dp}{dy} = -\frac{\Delta p}{l}$ if the laminar flow is fully developed. Now turn to cylindrical coordinates, we should readily have

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} = \frac{1}{\eta} \frac{dp}{dy}$$

(23)

Remember that $v = v(r)$, then $\frac{1}{r} \frac{d}{dr} (r \frac{dv}{dr}) = -\frac{1}{\eta} \frac{\Delta p}{r}$. Thus $r \frac{dv}{dr} = -\frac{1}{\eta} \frac{\Delta p}{r} r^2 + C_1$, and

$$v = -\frac{\Delta p}{4\eta l} r^2 + C_1 \ln r + C_2.$$

Because $v|_{r=0}$ must be finite, we have $C_1 = 0$. Also, $v|_{r=r_0} = 0$, we have $C_2 = \frac{\Delta p}{4\eta l} r_0^2$. Finally, we have $v = -\frac{\Delta p}{4\eta l} (r_0^2 - r^2)$. Rate of flow is then $q = \int vdA = \int_0^{r_0} 2\pi r v dr = \frac{\pi \Delta p r_0^4}{8\eta l}$. At 25°C, the viscosity of water is $8.94 \times 10^{-4} Pa \cdot s$, the pressure gradient is then $1.4 \times 10^6 Pa/m$. The power required to deliver this water over a 10m distance is $\frac{\Delta p}{l} Lq = 1.4 \times 10^9 Watt$. 

![Graph of function v vs r]