1. (10 points)

$$\vec{\nabla} p = \rho \vec{g} \tag{1}$$

Ideal gas EOS: ($v = \frac{V}{m} = \frac{1}{\rho}$ specific volume, $R^* = \frac{R}{M}$ specific gas constant)

$$pv = \frac{p}{\rho} = R^* T \tag{2}$$

$$\left.\frac{dp}{dz}\right|_T = R^* T \frac{d\rho}{dz} = -\rho g \tag{3}$$

$$R^* T \ln \frac{\rho}{\rho_0} = -g(z - z_0) \tag{4}$$

$$\rho(z) = \rho_0 \exp \frac{g}{R^* T}(z_0 - z) \tag{5}$$

For ideal $N_2$ gas at $z = 100m$ and $T = 300K$, $\rho(z)/\rho(0) = 0.989$.
Isothermal bulk modulus $B_T = -v \left.\frac{\partial p}{\partial v}\right|_T = \rho \left.\frac{\partial p}{\partial \rho}\right|_T$.

$$\left.\frac{dp}{dz}\right|_T = \frac{\partial p}{\partial \rho}\frac{d\rho}{dz} = \frac{B_T}{\rho}\frac{d\rho}{dz} = -\rho g \tag{6}$$

$$\frac{1}{\rho(z)} - \frac{1}{\rho(0)} = \frac{g}{B_T}(z - z_0) \tag{7}$$

For water at $z = -100m$, $\rho(z)/\rho(0) = 1.00045$.
2. (5 points)
For the derivation, please refer to *L.L., Fluid Mechanics, section 4.*

$$-\frac{dT}{dz} < \frac{g}{c_p} \tag{8}$$

where $c_p = \frac{7}{5}c_v$ and $c_v = \frac{5}{2}R/M$. Critical size of the temperature gradient is thus $-\frac{dT}{dz} < 0.0094K/m$.
Also, the derivation given in class is summarized below:
Let there be a temperature gradient dT/dz in an ideal gas under gravity. We know that hydrostatics requires $dP/dz = -\rho g$ for this gas. We know that there will be a convective instability if a piece of gas, lifted adiabatically, finds its density greater than that of the gas it displaced. Or, the gradient of temperature is subject to convective instability if

$$\frac{1}{V}\left(\frac{\partial V}{\partial z}\right)_{\text{adiabatic}} > \frac{1}{V}\left(\frac{\partial V}{\partial z}\right)_{\text{actual}}, \tag{9}$$

where "actual" means the gradient of the system without any imposed displacement. This latter can be written

$$\frac{1}{V}\left(\frac{\partial V}{\partial z}\right)_{\text{actual}} = \frac{1}{V}\left(\frac{\partial V}{\partial z}\right)_T + \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P \left(\frac{\partial T}{\partial z}\right)_{\text{actual}}. \tag{10}$$

For the adiabatic case, use $pV^\gamma = \text{const}$ and the equation of hydrostatics to write

$$\frac{1}{V}\left(\frac{\partial V}{\partial z}\right)_{\text{adiabatic}} = \frac{\rho g}{\gamma P}. \tag{11}$$

For the "actual" case, we need the ideal gas result

$$\frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P = \frac{1}{T}. \tag{12}$$

This enables the condition for instability to be written

$$\frac{\rho g}{\gamma P} > \frac{\rho g}{P} + \frac{1}{T}\left(\frac{\partial T}{\partial z}\right)_{\text{actual}}. \tag{13}$$

Rearranging this equation, the condition for convective instability is

$$-\left(\frac{\partial T}{\partial z}\right)_{\text{actual}} > \left(1 - \frac{1}{\gamma}\right)\frac{\rho g T}{P}. \tag{14}$$

The derivation above is the one given in class. Landau and Lifshitz solve the problem more generally in section 4 of *Fluid Mechanics.* Their result does not assume an ideal gas, and gives the condition for instability as

$$-\left(\frac{\partial T}{\partial z}\right)_{\text{actual}} > \frac{\rho g T}{C_p V}\left(\frac{\partial V}{\partial T}\right)_P. \tag{15}$$

If you work this out for an ideal gas, you recover the same result.
Numerically, the problem asks "Consider a large, thermally insulated tank of $N_2$ gas, under gravity, at $P_0$= 1 atm and $T_0$ = 300K, with a uniform temperature gradient $dT/dz < 0$. ... You can treat $N_2$ as a diatomic ideal gas, with $C_P/C_V = \gamma = 7/5$." The numerical answer is that instability occurs when $-dT/dz > 0.0094$ K/m.
3. (15 points)
Equation for the velocity potential is Laplace equation assuming irrotational flow:

$$\vec{\nabla}^2\Phi = 0 \tag{16}$$

Boundary conditions:

$$\left(\frac{\partial \Phi}{\partial z} + \frac{1}{g}\frac{\partial^2\Phi}{\partial t^2}\right)\bigg|_{z=0} = 0 \tag{17a}$$

$$v_z|_{z=-h} = \frac{\partial \Phi}{\partial z}\bigg|_{z=-h} = 0 \tag{17b}$$

$$\Phi = A\cos(kx - \omega t)\cosh k(z+h) \tag{18}$$

$$\omega^2 = gk\tanh kh \tag{19}$$

It can be easily seen that in the limiting cases $\lambda \ll h$ and $\lambda \gg h$, $\omega^2 = gk$ and $\omega = \sqrt{gh}k$ respectively.

$$v_g = \frac{\partial\omega}{\partial k} = \frac{1}{2}\sqrt{\frac{g\tanh kh}{k}} + \frac{gkh}{2\sqrt{gk\tanh kh}\cosh^2 kh} \tag{20a}$$

$$v_p = \frac{\omega}{k} = \sqrt{\frac{g}{k}\tanh kh} \tag{20b}$$

Group velocity is shown below:

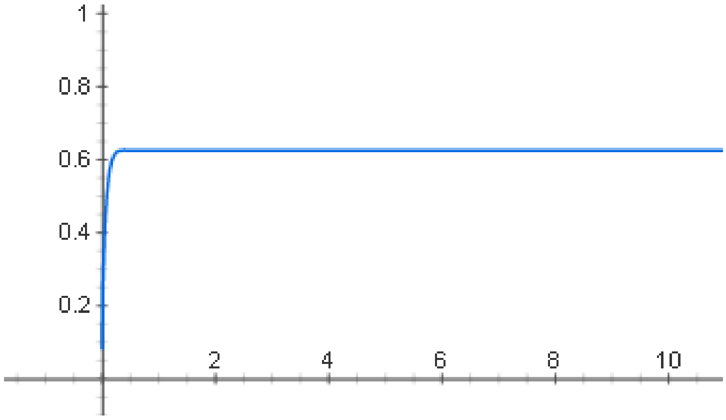
4. (15 points)
N-S equation reads:

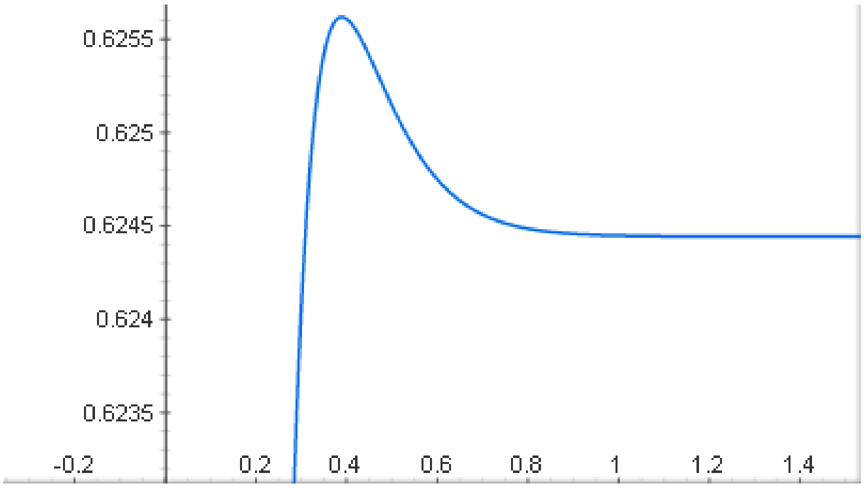
$$\vec{f} - \vec{\nabla}p + \eta\vec{\nabla}^2\vec{v} = \rho\left[\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right] \tag{21}$$

Assume steady laminar flow along the axis of the cylinder $\hat{y}$, then $v_y = v, v_x = v_z = 0, \frac{\partial v_y}{\partial t} = 0$. Ignore the effect of gravity, we thus have $\vec{f} = 0$. From the continuity equation, we have (for imcompressible fluid) $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$, and thus $\frac{\partial v_y}{\partial y} = 0$. N-S equations reduce to

$$\frac{\partial p}{\partial y} - \eta\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial z^2}\right) = 0 \tag{22a}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{1}{\rho}\frac{\partial p}{\partial z} = 0 \tag{22b}$$




From (22b), we know that $p = p(y)$, thus $\frac{\partial p}{\partial y} = \frac{dp}{dy}$, and we can further assume $\frac{dp}{dy} = -\frac{\Delta p}{l}$ if the laminar flow is fully developed. Now turn to cylindrical coordinates, we should readily have

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = \frac{1}{\eta}\frac{dp}{dy} \tag{23}$$

Remember that $v = v(r)$, then $\frac{1}{r}\frac{d}{dr}(r\frac{dr}{dv}) = -\frac{1}{\eta}\frac{\Delta p}{l}$. Thus $r\frac{dv}{dr} = -\frac{1}{\eta}\frac{\Delta p}{l}\frac{r^2}{2} + C_1$, and $v = -\frac{\Delta p}{4\eta l}r^2 + C_1 \ln r + C_2$. Because $v|_{r=0}$ must be finite, we have $C_1 = 0$. Also, $v|_{r=r_0} = 0$, we have $C_2 = \frac{\Delta p}{4\eta l}r_0^2$. Finally, we have $v = \frac{\Delta p}{4\eta l}(r_0^2 - r^2)$. Rate of flow is then $q = \int v dA = \int_0^{r_0} 2\pi r v dr = \frac{\pi \Delta p}{8\eta l}r_0^4$. At $25°C$, the viscosity of water is $8.94 \times 10^{-4} Pa \cdot s$, the pressure gradient is then $1.4 \times 10^6 Pa/m$. The power required to deliver this water over a $10m$ distance is $\frac{\Delta p}{l}Lq = 1.4 \times 10^5 Watt$.