1a. (3 points)

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{qB}{2}(x\dot{y} - y\dot{x})$$

$$m\ddot{x} = qB\dot{y} \qquad m\ddot{y} = -qB\dot{x} \qquad m\ddot{z} = 0$$

$$m\frac{d\vec{v}}{dt} = q\vec{v} \times \vec{B}$$

1b. (3 points)

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{x}_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} + q\frac{d}{dt}\frac{\partial}{\partial \dot{x}_i}\left[\vec{\nabla}\varphi(\vec{r}) \cdot \vec{v}\right] = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} + q\frac{d}{dt}\frac{\partial \varphi}{\partial x_i}$$

$$\frac{\partial L'}{\partial x_i} = \frac{\partial L}{\partial x_i} + q\frac{\partial}{\partial x_i}\left[\vec{\nabla}\varphi(\vec{r}) \cdot \vec{v}\right] = \frac{\partial L}{\partial x_i} + q\left[\vec{\nabla}\frac{\partial \varphi}{\partial x_i} \cdot \vec{v}\right]$$

$$\frac{d}{dt}A = \frac{\partial}{\partial t}A + \vec{v} \cdot \vec{\nabla}A$$

$$\frac{\partial}{\partial t}\frac{\partial \varphi}{\partial x_i} = 0$$

$\therefore$ equation unchanged.

1c. (3 points)

$$R = \frac{mv}{qB} \approx 1 \times 10^{-4}\ m$$

1d. (3 points)

$$L = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2) + \frac{qB}{2}\rho^2\dot{\varphi}$$

1e. (3 points)

$$E = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$E = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2)$$

1f. (3 points)

$$E = \frac{1}{2m}\left((p_x + \frac{qB}{2}y)^2 + (p_y - \frac{qB}{2}x)^2 + {p_z}^2\right)$$

$$E = \frac{1}{2m}\left({p_\rho}^2 + (\frac{p_\varphi}{\rho} - \frac{qB}{2}\rho)^2 + {p_z}^2\right)$$

1g. (3 points)

$$\frac{\partial L}{\partial q_i} = 0 \;\rightarrow\; \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0 \;\rightarrow\; p_i = \frac{\partial L}{\partial \dot{q}_i} = const.$$

p_z conserved in Cartesian system

p_z, p_φ conserved in cylindrical system

2a. (3 points)

$$S = \frac{mD^2}{2T}$$

2b. (3 points)

$$x = R(1 - \cos(\omega t))$$

$$y = R\sin(\omega t)$$

$$z = \frac{D}{T}t$$

helix, R arbitrary, $\omega = \frac{qB}{m}$

2c. (3 points)

$$S = \int_0^T dt[\frac{1}{2}m\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{qB}{2}(x\dot{y} - y\dot{x})]$$

$$S = \int_0^T dt[\frac{1}{2}m\left(R^2\omega^2 + \frac{D^2}{T^2}\right) + \frac{qB}{2}R^2\omega(\cos(\omega t) - 1)] = \frac{mD^2}{2T}$$

Helixes with arbitrary radius are allowed paths as well as straight line.

3. Mechanical Similarity. Charged particle in a magnetic field. Suppose not just distance and time, but other quantities are scaled. What can you learn from this "mechanical similarity" about the behavior of the charged particle in a magnetic field?

Consider the Lagrangian:

$$L = \frac{1}{2} m\vec{v}^2 + \frac{qB}{2} r^2 \dot{\phi}$$

When distances are scaled by α and times by β, this becomes

$$L' = \frac{1}{2} m\left(\frac{\alpha^2}{\beta^2}\right)\vec{v}^2 + \frac{qB}{2}\left(\frac{\alpha^2}{\beta}\right) r^2 \dot{\phi}$$

This shows that distance scaling without time scaling gives similarity. This agrees with our known view that circular orbits exist with any radius, but velocity and frequency have their ratio altered by a factor α. The most physical new thing to scale is magnetic field, $B \to \gamma B$. Then

$$L'' = \frac{1}{2} m\left(\frac{\alpha^2}{\beta^2}\right)\vec{v}^2 + \frac{qB}{2}\left(\frac{\gamma\alpha^2}{\beta}\right) r^2 \dot{\phi}$$

Now we see that if $\gamma = 1/\beta$, then time can also be scaled inversely with field. The frequency will go linearly with B. If instead of $B \to \gamma B$, we used $q \to \gamma q$ or $m \to m/\gamma$, we would also get time scaling, and the frequency would scale with q and inversely with m. So we learn that the cyclotron frequency is $\omega_c = qB/m$.

Mechanical similarity is just a nice version of dimensional analysis.