1a. (5 points)

$$x(U) = \frac{1}{2\pi\sqrt{2m}}\int_0^U \frac{T(E)dE}{\sqrt{U-E}}$$

$$x(U) = \frac{T}{\pi\sqrt{2m}}\sqrt{U}$$

$$U(x) = \frac{2\pi^2 m}{T^2}x^2$$

1b. (5 points)

$$\frac{t'}{t} = (\frac{l'}{l})^{1-\frac{1}{2}k}$$

$$\frac{E'}{E} = (\frac{l'}{l})^k = (\frac{t'}{t})^{\frac{2k}{2-k}}$$

$$\frac{1}{2}\frac{2k}{2-k} = 1 \quad k = 1$$

$$\therefore U(\alpha x) = \alpha U(x) \quad U(x) \sim x.$$

1c. (5 points)

$$x(U) = \frac{1}{2\pi\sqrt{2m}}\int_0^U \frac{T(E)dE}{\sqrt{U-E}}$$

$$x(U) = \frac{\beta}{2\pi\sqrt{2m}}\int_0^U \frac{E^{1/2}dE}{\sqrt{U-E}} = \frac{\beta U}{4\sqrt{2m}}$$

$$U(x) = \frac{4\sqrt{2m}}{\beta}|x|$$

2a. (5 points)

$$r = \frac{p}{1+e\cos(\varphi)}$$

$$\langle r\rangle = \int_0^{2\pi} \frac{pd\varphi}{1+e\cos(\varphi)} / \int_0^{2\pi} d\varphi$$

$$\langle r \rangle = \frac{p}{\sqrt{1-e^2}} \quad p = 38.1846 \text{ a.u.}$$

$$r_{min} = \frac{p}{1+e} = 30.543 \text{ a.u.}$$

$$r_{max} = \frac{p}{1-e} = 50.926 \text{ a.u.}$$

2b. (5 points)

$$a = \frac{p}{1-e^2} = 40.735 \text{ a.u.}$$

$$E = -\frac{\alpha}{2a} = -\frac{GMm}{2a} = 3.269 \times 10^{31} \text{ J}$$

2c. (5 points)

$$b = \frac{p}{\sqrt{1-e^2}} = \langle r \rangle = 39.439 \text{ a.u.}$$

$$L = b\sqrt{2m|E|} = 8.258 \ \times 10^{40} \text{ kg} \cdot \text{m}^2/\text{s}$$

$$T = \pi GMm\sqrt{m/2|E|^3} = 8.197 \times 10^9 \text{ seconds} = 259.9 \text{ years}$$

2d. (5 points)

$$\frac{K_{sun}}{K_{Pluto}} = \frac{\frac{1}{2}Mv_{sun}^2}{\frac{1}{2}mv_{Pluto}^2} = \frac{Mr_{sun}^2\omega^2}{mr_{Pluto}^2\omega^2}$$

$$Mr_{sun} = mr_{Pluto}$$

$$\frac{K_{sun}}{K_{Pluto}} = \frac{m}{M} = 1.51 \ \times 10^{-6}$$