\[ \theta = 2(\beta - \alpha) \]

Notice that the bonding is \( \beta - \alpha \) at each interface.

\[ v_{\text{reduced}} = \frac{r_0}{\rho} \sin \alpha = \frac{1}{n} \sin \alpha \]

\[ \theta = 2(\beta - \alpha) \] (full credit for the two equations)

\[ \sin \alpha = \frac{r}{r_0} \]

\[ \sin \beta = \frac{r_0}{\rho} \]

\[ \sin \alpha = \frac{1}{n} \sin \alpha \]

Conservation Laws

\[ mv_{\text{reduced}}^2 = m v_{\text{reduced}} v_0 \]

\[ \frac{1}{2} mv_{\text{reduced}}^2 = \frac{1}{2} mv^2 + U_0 \]

\[ \sin \alpha \sin \beta = \frac{m v_{\text{reduced}}^2}{v_0^2} = \frac{v_0^2}{v_{\text{reduced}}^2} = \sqrt{1 - \frac{U_0}{\frac{1}{2} mv_{\text{reduced}}^2}} = \frac{1}{n} < 1 \]

Note that \( \sin \beta = \frac{1}{n} \sin \alpha \) only works if \( \sin \alpha = \frac{r}{r_0} \frac{1}{n} \)

Beyond that point, there is no solution for \( \beta \).

Total external reflection: \( \theta = \pi - 2\alpha \)

For \( \frac{r_0}{\rho} < n \), \( \theta = 2(\beta - \alpha) \) and \( 2\beta < \pi \)

For \( \frac{r_0}{\rho} > n \), \( \theta = \pi - 2\alpha \) and \( 2\beta \), having reached \( \pi \), steps existing

For \( \frac{r_0}{\rho} < n \),

\[ \frac{\sin \beta}{\sin \alpha} = \frac{\sin \alpha + \frac{\theta}{2}}{\sin \alpha} = \cos \frac{\theta}{2} + \frac{1}{\tan \alpha} \frac{\sin \theta}{2} = \frac{1}{n} \]

\[ \frac{1}{\tan \alpha} = \sqrt{\frac{a_z^2}{a_{z_0}^2}} - 1 = \frac{1}{n} - \cos \theta/2 \]

\[ \frac{a_z^2}{a_{z_0}^2} = \frac{1}{1 + \left(\frac{n - \cos \theta/2}{\sin \theta/2}\right)^2} \]
But for $\frac{p}{a} > n$, \( \theta = \pi - 2 \sin^{-1} \frac{p}{a} \)
\( \frac{p}{a} = \sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \cos \frac{\theta}{2} \)

To summarize, \[
\left(\frac{p}{a}\right)^2 = \frac{1}{1 + \left(\frac{\sqrt{n} \cos \frac{\theta}{2}}{\sin \frac{\theta}{2}}\right)^2} \quad \text{if} \quad \frac{p}{a} < n < 1
\]
\[
\left(\frac{p}{a}\right)^2 = \cos^2 \frac{\theta}{2} \quad \text{if} \quad \frac{p}{a} > n
\]

and these two expressions go continuously to \( \left(\frac{p}{a}\right)^2 = n^2 \) as \( \cos \frac{\theta}{2} \rightarrow n \)

* Full credit for (b) requires only this.

(d) If \( n = \sqrt{2} \), \( \alpha_{critical} = \sin^{-1} \frac{1}{\sqrt{2}} = \frac{\pi}{4} \)
at this point, \( \sin \beta = 1 \), \( \beta = \frac{\pi}{2} \) and \( \theta = 2 \left(\frac{\pi - \frac{\pi}{4}}{2}\right) = \frac{\pi}{2} \)
\( \rho / a = \sin \alpha_{c} = n = \frac{1}{\sqrt{2}} \)

\[\begin{align*}
\theta &\rightarrow +\infty \\
\rho / a &\rightarrow \frac{1}{\sqrt{2}} \\
1 / \rho / a &\rightarrow +\infty, \theta \rightarrow 0 \\
\end{align*}\]
2a. (10 points)

\[ L = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} M \dot{x}_2^2 + \frac{1}{2} m \dot{x}_3^2 - \frac{1}{2} K (x_2 - x_1)^2 - \frac{1}{2} K (x_3 - x_2)^2 \]

\[ M \ddot{X} + KX = 0 \]

\[ M = \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix}, \quad K = \begin{pmatrix} K & -K & 0 \\ -K & 2K & -K \\ 0 & -K & K \end{pmatrix} \]

\[ \det(-\omega^2 M + KX) = 0 \]

\[ \omega_1 = 0, A_1 = \frac{1}{\sqrt{3}} (1 \quad 1 \quad 1)^T \]

\[ \omega_2 = \frac{\sqrt{K}}{\sqrt{m}}, A_2 = \frac{1}{\sqrt{2}} (1 \quad 0 \quad -1)^T \]

\[ \omega_3 = \frac{\sqrt{K + 2K}}{\sqrt{m}}, A_3 = \frac{1}{\sqrt{2 + 4 \frac{m^2}{M^2}}} \begin{pmatrix} 1 & -2 \frac{m}{M} & 1 \end{pmatrix}^T \]

2b. (5 points)

\( A_3 \) is responsible for greenhouse effect. Parity selection rule states that the excited vibration mode must have odd parity.