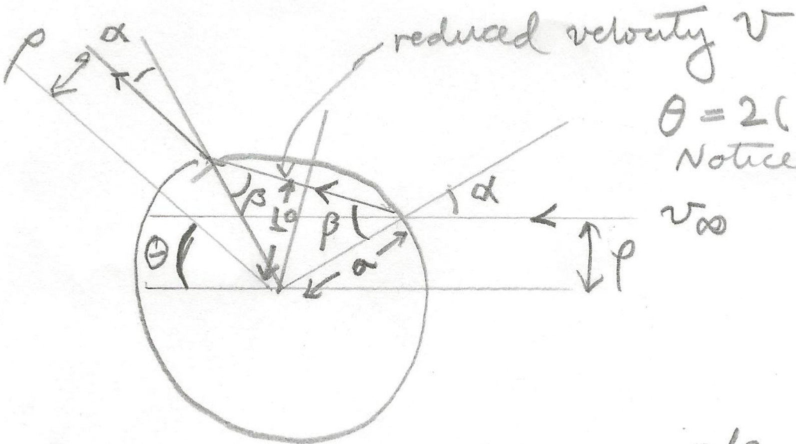
HW5 #1

(20 points)



$\theta = 2(\beta - \alpha)$
Notice that the bending is $\beta-\alpha$ at each interface.

Trigonometry

$\sin\alpha = \rho/a$

$\sin\beta = r_0/a = \frac{r_0}{\rho}\sin\alpha = \frac{1}{n}\sin\alpha$

$\boxed{\theta = 2(\beta - a)}$

full credit for (a) requires these 2.

Conservation Laws

$m v_\infty \rho = m v r_0$

$\frac{1}{2} m v_\infty^2 = \frac{1}{2} m v^2 + U_0$

$\frac{\sin\alpha}{\sin\beta} = n = \frac{\rho}{r_0} = \frac{v}{v_\infty} = \boxed{\sqrt{1 - \frac{U_0}{\frac{1}{2} m v_\infty^2}} = n} < 1$

Note that $\sin\beta = \frac{1}{n}\sin\alpha$ only works if $\sin\alpha = \frac{\rho}{a} \le n$

Beyond that point, there is no solution for β.

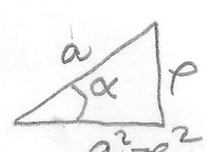
Total external reflection. $\theta = \pi - 2\alpha$

For $\frac{\rho}{a} < n$, $\theta = 2(\beta - \alpha)$ and $2\beta < \pi$

for $\frac{\rho}{a} > n$ $\theta = \pi - 2\alpha$ and 2β, having reached π, stops existing

For $\rho/a < n$,

$\frac{\sin\beta}{\sin\alpha} = \frac{\sin\left(\alpha + \frac{\theta}{2}\right)}{\sin\alpha} = \cos\frac{\theta}{2} + \frac{1}{\tan\alpha}\sin\frac{\theta}{2} = \frac{1}{n}$



$\frac{1}{\tan\alpha} = \sqrt{\frac{a^2}{\rho^2} - 1} = \frac{\frac{1}{n} - \cos\theta/2}{\sin\theta/2}$

so $\frac{\rho^2}{a^2} = \frac{1}{1 + \left(\frac{1/n - \cos\theta/2}{\sin\theta/2}\right)^2}$

But for $\frac{\rho}{a} > n$, $\quad \theta = \pi - 2\sin^{-1}\frac{\rho}{a}$

$$\frac{\rho}{a} = \sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \cos\frac{\theta}{2}$$

To summarize

$$* \quad \left(\frac{\rho}{a}\right)^2 = \frac{1}{1 + \left(\frac{1/n - \cos\theta/2}{\sin\theta/2}\right)^2} \quad \text{if } \frac{\rho}{a} < n < 1$$

$$\left(\frac{\rho}{a}\right)^2 = \cos^2\frac{\theta}{2} \quad \text{if } 1 > \frac{\rho}{a} > n$$

and these two expressions go continuously to $\left(\frac{\rho}{a}\right)^2 = n^2$ as $\cos\frac{\theta}{2} \to n$

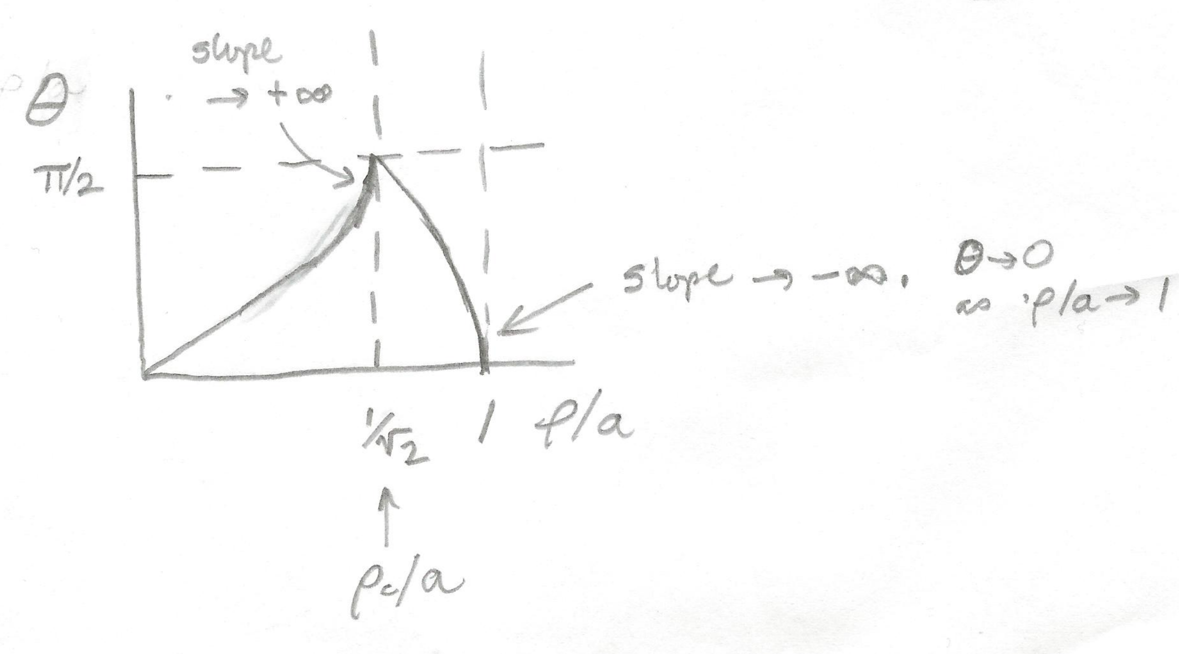
* full credit for (b) requires only this.

(d) if $n = 1/\sqrt{2}$, $\quad \alpha_{\text{critical}} = \sin^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}$

at this point $\sin\beta = 1$, $\beta = \pi/2$

and $\theta_c = 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \pi/2$

$\rho_c/a = \sin\alpha_c = n = \frac{1}{\sqrt{2}}$



2a. (10 points)

$$L = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}M\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2 - \frac{1}{2}K(x_2 - x_1)^2 - \frac{1}{2}K(x_3 - x_2)^2$$

$$\boldsymbol{M}\ddot{X} + \boldsymbol{K}X = 0$$

$$\boldsymbol{M} = \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix} \ \boldsymbol{K} = \begin{pmatrix} K & -K & 0 \\ -K & 2K & -K \\ 0 & -K & K \end{pmatrix}$$

$$\det(-\omega^2\boldsymbol{M} + \boldsymbol{K}X) = 0$$

$$\omega_1 = 0, A_1 = \frac{1}{\sqrt{3}}(1 \quad 1 \quad 1)^T$$

$$\omega_2 = \sqrt{\frac{K}{m}}, A_2 = \frac{1}{\sqrt{2}}(1 \quad 0 \quad -1)^T$$

$$\omega_3 = \sqrt{\frac{K}{m} + \frac{2K}{M}}, A_3 = \frac{1}{\sqrt{2 + 4\frac{m^2}{M^2}}}\left(1 \quad -2\frac{m}{M} \quad 1\right)^T$$

2b. (5 points)

A_3 is responsible for greenhouse effect. Parity selection rule states that the excited vibration mode must have odd parity.