Physics 501 solutions to HW7

problem 1. (5 points)

$$\dot{\vec{\phi}} = \dot{\phi}\hat{z} \tag{1a}$$

$$\dot{\vec{\theta}} = \dot{\theta}\cos\phi\hat{x} + \dot{\theta}\sin\phi\hat{y} \tag{1b}$$

$$\dot{\vec{\psi}} = \dot{\psi}\cos\theta\hat{z} + \dot{\psi}\sin\theta\sin\phi\hat{x} - \dot{\psi}\sin\theta\cos\phi\hat{y} \tag{1c}$$

hence

$$\Omega_x = \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi \tag{2a}$$

$$\Omega_y = \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi \tag{2b}$$

$$\Omega_z = \dot{\phi} + \dot{\psi}\cos\theta \tag{2c}$$

problem 2. (5 points + 5 points) **See next page.**

problem 3. (10 points)

$$I_1\tfrac{d\Omega_1}{dt} + (I_3 - I_2)\Omega_2\Omega_3 = 0 \tag{3a}$$

$$I_2\tfrac{d\Omega_2}{dt} + (I_1 - I_3)\Omega_3\Omega_1 = 0 \tag{3b}$$

$$I_3\tfrac{d\Omega_3}{dt} + (I_2 - I_1)\Omega_1\Omega_2 = 0 \tag{3c}$$

$$\Omega_1\tfrac{d\Omega_1}{dt} = \tfrac{I_2-I_3}{I_1}\Omega_1\Omega_2\Omega_3 \tag{4a}$$

$$\Omega_2\tfrac{d\Omega_2}{dt} = \tfrac{I_3-I_1}{I_2}\Omega_1\Omega_2\Omega_3 \tag{4b}$$

$$\Omega_3\tfrac{d\Omega_3}{dt} = \tfrac{I_1-I_2}{I_3}\Omega_1\Omega_2\Omega_3 \tag{4c}$$

$$\frac{1}{2}\frac{d}{dt}(\vec{\Omega}\cdot\vec{\Omega}) = \frac{\Omega_1\Omega_2\Omega_3}{I_1I_2I_3}[(I_1 - I_2)I_3^2 + (I_2 - I_3)I_1^2 + (I_3 - I_1)I_2^2] \tag{5}$$

For a symmetric top ($I_1 = I_2 \neq I_3$), we already have $\frac{1}{2}\frac{d}{dt}(\vec{\Omega}\cdot\vec{\Omega}) = 0$. For a rotator ($I_1 = I_2 = I = \Sigma m x_3^2$, $I_3 = 0$),

$$I\tfrac{d\Omega_1}{dt} - I\Omega_2\Omega_3 = 0 \tag{6a}$$

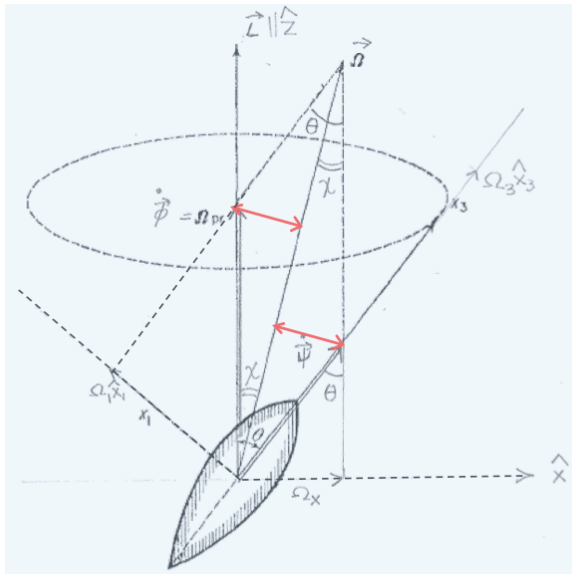
$$I\tfrac{d\Omega_2}{dt} + I\Omega_3\Omega_1 = 0 \tag{6b}$$

We still have $\frac{1}{2}\frac{d}{dt}(\vec{\Omega}\cdot\vec{\Omega}) = 0$

2. Define the earth to be a symmetric top, and the "north pole" to mean the symmetry axis (**x**3) of the earth. It is inclined by an angle θ relative to the angular momentum vector **L** of the earth. The angular velocity vector $\vec{\Omega}$ is inclined by an angle χ relative to the angular momentum vector **L**. The deviation $(I_3-I_1)/I_1$ from spherical symmetry of the earth is 0.0033. The earth's angular velocity vector, projected onto the earth's surface, is about 5 m from the north pole. The earth's radius is about $R_e=6.37 \times 10^6$m.

a. For free rotations of a symmetric top, show that $\dot{\phi}\sin\chi = \dot{\psi}\sin(\theta - \chi)$.

Answer: This is just the statement that the red lines on the drawing to the right have equal length. That follows because the parallelogram defined by the vectors $\vec{\dot{\phi}}$ and $\vec{\dot{\psi}}$ is split into two equivalent triangles by the $\vec{\Omega}$ vector.



b. How large is χ for the earth, and how far from the angular velocity vector is the angular momentum vector of the earth (both vectors considered just as points near the pole where they intersect the earth's surface.)

Answer: The earth's angular velocity (1 revolution per day) is Ω_3. The component $\vec{\dot{\psi}}$ of this is small, $\dot{\psi} = (1 - I_3/I_1)\Omega_3 = -0.0033\Omega_3$. The minus sign tells us that χ is negative – the $\vec{\Omega}$ vector tilts to the left of the $\vec{M}$ vector (the space Z axis.) The remaining part, $1.0033\Omega_3$, comes from the $\vec{\dot{\phi}}$ - component, $\dot{\phi}\cos\theta$. We are given $|\theta - \chi| = 5\text{m}/R_e \approx 0.8 \times 10^{-6}\text{rad}$. We can calculate χ by using $\dot{\phi}\sin\chi = \dot{\psi}\sin(\theta - \chi) \approx (0.0033\Omega_3)(\theta - \chi)$ (using the small angle approximation for the sine.) Since $\dot{\phi}\cos\theta = 1.0033\Omega_3 = \dot{\phi}\cos(\theta - \chi + \chi) \approx \dot{\phi}\cos\chi$ (this follows from $\theta - \chi$ being very small), we can divide the formula for $\dot{\phi}\sin\chi$ by the formula for $\dot{\phi}\cos\chi$, obtaining $\tan\chi = (0.0033\Omega_3)(\theta - \chi)/1.0033\Omega_3$, which tells us that $\chi \approx (0.0033)(\theta - \chi) \approx 3 \times 10^{-9}$ radians. Projected to the surface of earth, the angle χ subtends $(0.0033)(5\text{m}) \approx 1.7\text{cm}$. These are all remarkably small numbers. The angular precession of the rigid earth is very hard to observe.