1. (8 points) This simple illustration is from WIKIPEDIA. Set the direction of gravity to be $-\hat{z}$ in the local frame of rotation, and the direction of earth rotation to be \hat{Z} . If the angle between \hat{z} and \hat{Z} is θ , the latitude is then $\frac{\pi}{2} - \theta$. Since the direction of the centrifugal force is along \hat{z} , only Coriolis force and the usual pendulum restoring force are exerted in the x-y plane. Let the angular velocity of earth rotation to be Ω , and that of oscillation of pendulum to be ω , then

$$\frac{d^2x}{dt^2} = -\omega^2 x + 2\Omega \frac{dy}{dt} \sin\theta \tag{1a}$$

$$\frac{d^2y}{dt^2} = -\omega^2 y - 2\Omega \frac{dx}{dt} \sin\theta \tag{1b}$$

Using complex coordinates z = x + iy, above equations read

$$\frac{d^2z}{dt^2} + 2i\Omega\frac{dz}{dt}\sin\theta + \omega^2 z = 0 \tag{2}$$

To first order this equation has the solution

$$z = e^{-i\Omega t \sin\theta} (c_1 e^{i\omega t} + c_2 e^{-i\omega t}) \tag{3}$$

Terms in the parenthesis describe the motion of the pendulum in the x-y plane, while the prefactor gives the rotation of the oscillation plane. (Without directly looking at the evolving of complex vector with time, you can just easily see the rotation of the pendulum plane if you set $c_1 = c_2$.) At the equator, the oscillation plane remains fixed relative to the earth. At the North/South Pole, the pendulum rotates 2π per day.

2. (20 points)

$$L = \frac{1}{2}m\mathbf{v}^2 + q\mathbf{v}\cdot\mathbf{A} - U(\mathbf{r}) \tag{4}$$

$$p_i = \frac{\partial L}{\partial v_i} = mv_i + qA_i \tag{5}$$

$$H = \mathbf{p} \cdot \mathbf{v} - L = \frac{1}{2m} (\mathbf{p} - q\mathbf{A})^2 + U(\mathbf{r})$$
(6)

$$\dot{r_i} = \frac{1}{m}(p_i - qA_i) \tag{7a}$$

$$\dot{p_i} = \frac{q}{m} (\mathbf{p} - q\mathbf{A}) \cdot \frac{\partial \mathbf{A}}{\partial r_i} - \frac{\partial U}{\partial r_i}$$
(7b)

$$m\ddot{r}_{i} = \dot{p}_{i} - q\dot{A}_{i} = \frac{q}{m}(\mathbf{p} - q\mathbf{A}) \cdot \frac{\partial \mathbf{A}}{\partial r_{i}} - \frac{\partial U}{\partial r_{i}} - q\dot{A}_{i}$$

$$\tag{8}$$

total derivative $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla)$

$$m\ddot{r}_i = q\mathbf{v} \cdot \frac{\partial \mathbf{A}}{\partial r_i} - \frac{\partial U}{\partial r_i} - (\mathbf{v} \cdot \nabla)\mathbf{A} - q\frac{\partial \mathbf{A}}{\partial t}$$
(9)

if $\mathbf{A} = \mathbf{A}(\mathbf{r}), \ \frac{\partial \mathbf{A}}{\partial t} = 0$

$$m\ddot{\mathbf{r}} = q\mathbf{v} \times (\nabla \times \mathbf{A}) - \nabla U \tag{10}$$

if $\mathbf{A} = \mathbf{A}(\mathbf{r}, t)$, and $U(\mathbf{r}, t) = q\Phi(\mathbf{r}, t)$,

$$m\ddot{\mathbf{r}} = q\mathbf{v} \times (\nabla \times \mathbf{A}) - q\frac{\partial \mathbf{A}}{\partial t} - q\nabla\Phi$$
(11)

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} \tag{12a}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{12b}$$

$$m\ddot{\mathbf{r}} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B} \tag{13}$$

3. (2+10 points)

We have $r_o = r$ and $\phi_0 = \phi - \theta$. $L = \frac{1}{2}m\vec{v_0}^2 - U(r,\phi) = \frac{1}{2}m\vec{v_0}^2 - U(r_0,\phi_0 + \theta(t))$. Although the potential energy U depends on coordinate, equation of motion will remain unchanged. In the lab frame, if there is a potential field $U(r_0,\phi_0)$, the Lagrangian is $L_0 = \frac{1}{2}m\vec{v_0}^2 - U(r_0,\phi_0)$. In the rotating frame, the force is just the lab frame force but rotated with time. So the gradient by (r,ϕ) of $U(r,\phi) = U(r_0,\phi_0+\theta)$ is also the gradient by $(r_0,\phi_0+\theta)$.

$$\vec{v_0}^2 = \dot{r_0}^2 + r_0^2 (\dot{\phi} - \dot{\theta})^2 = \dot{r}^2 + r^2 (\dot{\phi} - \dot{\theta})^2 \tag{14}$$

$$L_{rot} = \frac{1}{2}m\vec{v_0}^2 - U(r,\phi) = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2(\dot{\phi} - \dot{\theta})^2 - U(r,\phi)$$
(15)

$$p_r = m\dot{r} \tag{16a}$$

$$p_{\phi} = mr^2(\dot{\phi} - \dot{\theta}) \tag{16b}$$

$$H_{rot} = p_r \dot{r} + p_\phi \dot{\phi} - L_{rot} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - \frac{1}{2}mr^2\dot{\theta}^2 + U(r,\phi)$$
(17)

$$H_{lab} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2(\dot{\phi} - \dot{\theta})^2 + U(r,\phi)$$
(18)

$$H_{rot} - H_{lab} = mr^2 \dot{\theta} (\dot{\phi} - \dot{\theta}) = p_{\phi} \dot{\theta}$$
⁽¹⁹⁾

when $\dot{\theta} = 0$, lab frame and rotating frame only differ by a constant angle, thus the Hamiltonians are equal. 4. (10 points)



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