

1. (5 points)

$$H = \frac{p^2}{2m} + mgq \quad (1)$$

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \quad (2a)$$

$$\dot{p} = -\frac{\partial H}{\partial q} = -mg \quad (2b)$$

$$p(t) = p_0 - mgt \quad (3a)$$

$$q(t) = q_0 + \frac{p_0}{m}t - \frac{1}{2}gt^2 \quad (3b)$$

$$q = q_0 + \frac{\frac{p_0^2}{m} - p^2}{2m^2g} \quad (4)$$

2. (10 points)

$$x(t) = \frac{x_1 t_2 - x_2 t_1}{t_2 - t_1} + \frac{x_2 - x_1}{t_2 - t_1} t \quad (5a)$$

$$y(t) = -\frac{1}{2}gt^2 + \left(\frac{y_2 - y_1}{t_2 - t_1} + \frac{1}{2}g(t_1 + t_2) \right)t + \frac{y_1 t_2 - y_2 t_1}{t_2 - t_1} - \frac{1}{2}gt_1 t_2 \quad (5b)$$

$$S = \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} \left[\frac{1}{2}m(v_x^2 + v_y^2) - mg y \right] dt = S_{v_x} + S_{v_y} + S_y \quad (6)$$

$$S_{v_x} = \frac{1}{2}m \frac{(x_2 - x_1)^2}{t_2 - t_1} \quad (7a)$$

$$S_{v_y} = \frac{1}{2}m \left[\frac{1}{3}g^2(t_2^3 - t_1^3) - g(t_2^2 - t_1^2) \left(\frac{y_2 - y_1}{t_2 - t_1} + \frac{1}{2}g(t_1 + t_2) \right) + \left(\frac{y_2 - y_1}{t_2 - t_1} + \frac{1}{2}g(t_1 + t_2) \right)^2 (t_2 - t_1) \right] \quad (7b)$$

$$S_y = -mg \left[-\frac{1}{6}g(t_2^3 - t_1^3) + \left(\frac{y_2 - y_1}{t_2 - t_1} + \frac{1}{2}g(t_1 + t_2) \right) \frac{1}{2}(t_2^2 - t_1^2) + \left(\frac{y_1 t_2 - y_2 t_1}{t_2 - t_1} - \frac{1}{2}gt_1 t_2 \right)(t_2 - t_1) \right] \quad (7c)$$

$$S = \frac{1}{2}m \frac{(x_2 - x_1)^2}{t_2 - t_1} + \frac{1}{2}m \frac{(y_2 - y_1)^2}{t_2 - t_1} - \frac{1}{2}mg(y_1 + y_2)(t_2 - t_1) - \frac{1}{24}mg^2(t_2 - t_1)^3 \quad (8)$$

$$\frac{\partial S}{\partial x_1} = -m \frac{x_2 - x_1}{t_2 - t_1} = -p_{x_1} \quad (9a)$$

$$\frac{\partial S}{\partial y_1} = -m \frac{y_2 - y_1}{t_2 - t_1} - \frac{1}{2}mg(t_2 - t_1) = -p_{y_1} \quad (9b)$$

$$\frac{\partial S}{\partial x_2} = m \frac{x_2 - x_1}{t_2 - t_1} = p_{x_2} \quad (9c)$$

$$\frac{\partial S}{\partial y_2} = m \frac{y_2 - y_1}{t_2 - t_1} - \frac{1}{2}mg(t_2 - t_1) = p_{y_2} \quad (9d)$$

S serves as an F_1 generating function that transforms from the final point to the initial point. Reversely, -S transforms from the initial point to the final point.

3. (10 points)

$$dF_4 = -qdp + QdP + (H' - H)dt \quad (10)$$

$$H'(P, Q) = P = H = \frac{p^2}{2m} + mgq = E \quad (11)$$

$$dF_4 = -qdp + QdP \quad (12)$$

$$\dot{Q} = \frac{\partial H'}{\partial P} = 1, Q = t + C \quad (13)$$

$$\frac{\partial F_4}{\partial p} = -q = -\frac{P - p^2/2m}{mg} \quad (14)$$

$$F_4 = -\frac{pP}{mg} + \frac{p^3}{6m^2g} + f(P) \quad (15)$$

For simplicity, take $f(P) = 0$.

$$Q = \frac{\partial F_4}{\partial P} = -\frac{p}{mg} \quad (16)$$

$$q = \frac{P}{mg} - \frac{gQ^2}{2} = \frac{E}{mg} - \frac{g(t+C)^2}{2} \quad (17a)$$

$$p = -mgQ = -mg(t + C) \quad (17b)$$

4. (10 points)

The Hamilton-Jacobi equation:

$$\frac{\partial S}{\partial t} + \frac{1}{2m} \left(\frac{\partial S}{\partial x} \right)^2 + \frac{1}{2m} \left(\frac{\partial S}{\partial y} \right)^2 + mgy = 0 \quad (18)$$

$$S = S_x + S_y + S_t + S_0 \quad (19)$$

$$\frac{1}{2m} \left(\frac{\partial S_x}{\partial x} \right)^2 = \alpha_x \quad (20a)$$

$$\frac{1}{2m} \left(\frac{\partial S_y}{\partial y} \right)^2 + mgy = \alpha_y \quad (20b)$$

$$\frac{\partial S_t}{\partial t} = -(\alpha_x + \alpha_y) \quad (20c)$$

$$\frac{\partial S_x}{\partial x} = \sqrt{2m\alpha_x} \quad (21a)$$

$$\frac{\partial S_y}{\partial y} = \sqrt{2m(\alpha_y - mgy)} \quad (21b)$$

$$\frac{\partial S_t}{\partial t} = -(\alpha_x + \alpha_y) \quad (21c)$$

$$S_x = \sqrt{2m\alpha_x}x \quad (22a)$$

$$S_y = -\frac{1}{3m^2g} [(2m\alpha_y - 2m^2gy)^{3/2} - (2m\alpha_y)^{3/2}] \quad (22b)$$

$$S_t = -(\alpha_x + \alpha_y)t \quad (22c)$$

Now the generating function F_2 is just $S(x, y; \alpha_x, \alpha_y; t)$. Take $\alpha_{x,y}$ as the new momenta.

$$\beta_x = \frac{\partial S}{\partial \alpha_x} = \sqrt{\frac{m}{2\alpha_x}}x - t \quad (23a)$$

$$\beta_y = \frac{\partial S}{\partial \alpha_y} = -\frac{1}{mg} [\sqrt{2m\alpha_y - 2m^2gy} - \sqrt{2m\alpha_y}] - t \quad (23b)$$

$$x = \sqrt{\frac{2\alpha_x}{m}} (\beta_x + t) \quad (24a)$$

$$y = -\frac{1}{2}g(\beta_y + t)^2 + g(\beta_x + t)\sqrt{\frac{2\alpha_y}{mg^2}} \quad (24b)$$

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