Physics 501 – Classical Mechanics – Fall 2012 – HW #10 due Friday Dec. 7

1. Hydrostatics. Euler's equation for a fluid at rest under gravity is $\vec{\nabla} p = \rho \vec{g}$. **(a)** Derive an equation for $\rho(z)$ for an ideal gas at constant temperature. **(b)** What is the ratio $\rho(z)/\rho(0)$ for N_2 gas (considered ideal) at z = 100m, p_0 = 1 atm and T = 300K? **(c)** Derive an equation for $\rho(z)$ for a fluid at constant temperature with an isothermal bulk modulus B_T that is independent of pressure. **(d)** What is the value of $\rho(z)/\rho(0)$ for water (Bulk modulus 2.2 x 10^9 Pa) at z = -100m, p_0 = 1 atm and T = 300K?

2. **Absence of convection**. A fluid at rest under gravity can have a density gradient only in the vertical direction. This tells us that, if there is non-zero thermal expansion, then a temperature gradient can only be in the vertical direction. With no sources or sinks of thermal energy, the fluid will eventually equilibrate to constant temperature because of thermal conduction. But this is a fairly slow process, especially for gases, and mechanical stability does not forbid a varying T(z). A temperature gradient will induce a convective instability if denser fluid lies on top of less dense fluid. Actually, the instability occurs even when a less dense fluid is on top, provided the denser fluid below expands insufficiently while rising adiabatically, and fails to experience a downward restoring force, because it is now less dense than the fluid it displaced. Consider a large, thermally insulated tank of N_2 gas, under gravity, at p_0 = 1 atm and T_0 = 300K, with a uniform temperature gradient dT/dz < 0. Assume that thermal conduction is too slow to be relevant. Find the critical size of the temperature gradient, beyond which convection is likely to set in (give both a formula and a number). You can treat N_2 as a diatomic ideal gas, with $C_p/C_v = \gamma = 7/5$.

3. **Gravity waves**. In class we will work out the nature of gravity waves in an incompressible fluid of density ρ, whose stationary surface is at z=0 at pressure p_0, depth large compared to wavelength, and transverse dimensions very large. One of the main results is the dispersion relation $\omega^2=gk$. Repeat the derivation for the case of a fluid with depth h. Obtain the general dispersion relation, and the limiting cases $\lambda << h$ and $\lambda >> h$. Sketch with fair accuracy (i.e. real numbers) the velocity versus depth for water waves of wavelength 1m, as the depth varies from 0.1m to 10m.

4. **Poiseuille formula**. The rate of flow of fluid through a cylindrical pipe of radius R is proportional to inverse viscosity η^{-1}, density ρ, pressure gradient, and radius R^4. Derive the formula from the Navier-Stokes equation (this assumes that water is incompressible). How much pressure gradient is required to deliver 10 kg water/sec through a pipe of radius 2 mm? How much energy is required to deliver this water over a 10 m distance?