Physics 501 – Classical Mechanics – Fall 2012  

HW #2 due Friday Sept. 14

1. **Equation of a cycloid.** As done in lecture, use coordinates \((x,y)\) for the horizontal and vertical coordinates of the position vector of a point on the rolling wheel, and use \((0,0)\) as the starting point. For small \((x,y)\) show that the cycloid obeys the approximate formula \(y = C x^{2/3}\), and find the value of the coefficient \(C\).

2. **Velocity on the optimum Bernoulli path.** This was not part of the challenge problem, so perhaps it was not known to Bernoulli, or perhaps he thought it might somehow simplify the challenge to ask about time-evolution.

   (a) Show that the time evolution (of the particle of mass \(m\) falling on the inverted cycloid) corresponds to the point on the wheel moving at constant angular velocity \(\phi\), and find the value of this constant. **Note** - in case this seems counter-intuitive (as it did to me, at first), recall that the point where the wheel touches the ground is instantaneously stationary, even though \(\dot{\phi} \neq 0\). Also, note that this is not just for small time, but for the full evolution, up to a possible moment where the particle departs from the cycloid and continues horizontally (below the starting point by \(2r\)) with constant velocity.

   (b) Use this to show that the horizontal velocity on the optimal Bernoulli path increases quadratically in time, for small \(t\), and find the coefficient.

3. **"Optimum" path with least area and mean slope.** Consider paths from \((x_0,y_0) = (0,0)\) to \((x_1,y_1) = (a,1)\). Find the path that minimizes \(F[y(x)] = \int_{x_0}^{x_1} dx f(y(x),y'(x),x)\), where the function \(f(y,y',x) = y^2 + a^2 y'^2\). **Do this two ways:**

   (a) Use the Euler-Lagrange equation directly, getting a second order differential equation.

   (b) As done in class, show that, in place of the Euler-Lagrange equation, you can use instead a "first integral" of this equation, namely \(f - y \frac{\partial f}{\partial y'} = \text{const}\). This works only because \(f(y,y',x)\) does not depend explicitly on the independent variable \(x\). However, in this case, it doesn’t simplify the algebra! The first procedure gives a linear 2\(^{nd}\)-order equation which is not hard to solve, whereas the second gives a non-linear 1\(^{st}\)–order equation which is not any easier.

4. **Flexible pendulum.** A spring has length \(L\), spring constant \(K\), and negligible mass. It has one end fixed at the point \((x,y) = (0,0)\). With no mass attached, its other end rests at \((x,y) = (0,-L)\). A point mass \(m\) is attached at the lower end, and motion under gravity in the \((x,y)\) plane is observed. Use the usual circular coordinates \(r = (x,y) = r \sin \phi, -\cos \phi\) for the position of the mass. Unlike actual light-weight springs, this one does not flex, it can only stretch.

   (a) Write the Lagrangian.

   (b) Write the equations of motion for \(r\) and \(\phi\).

   (c) Note that there is a complicated force of constraint at the fixed point of the spring. Why, or under what circumstances, is it appropriate not to include that in the problem?