D. Richards – Advanced Mathematical Methods with Maple. – Cambridge U. Press (2002)

in the description of dynamical systems with parameters that vary periodically with time; for instance the vertical pendulum with a point of support that moves periodically, as in exercise 12.38 (page 475).

In order to provide some idea of how such systems can behave, we start this section with a description of two systems. Both examples are essentially vertical pendulums and in each case the length is made to vary periodically. With the correct choice of this period it is shown that the energy (that is the amplitude of the swing) can be made to increase rapidly. The method of supplying energy to a system whereby system parameters are varied periodically is known as *parametric pumping*.

12.3.2 Parametric resonance: the swing

The first example of parametric resonance with which you are bound to be familiar is a child on a playground swing. Very careful observation of the child shows that the amplitude of the motion is increased by the rhythmical bending and straightening of the child's body with the effect that the centre of mass is raised as the swing passes through its lowest point and lowered when the swing reaches its highest point. An idealisation of this motion is obtained by treating the swing and child as a vertical pendulum with shifts in the centre of mass taking place instantaneously at the lowest and highest points, as shown in the diagram. The advantage of this approximation is that we can understand the motion without solving any differential equations, although conservation of angular momentum and energy are needed.

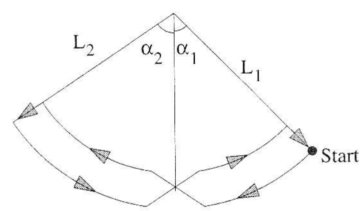


Figure 12.1

During each quarter oscillation, when the length L of the pendulum is constant, the swing behaves like the vertical pendulum for which the energy is

$$E = \frac{1}{2}mL^2\dot\theta^2 - mgL\cos\theta,$$

where θ is the angle between the swing and the downward vertical. Suppose that the system is released from rest at an angle $\theta = \alpha_1$ with length L_1, then the angular velocity $\dot\theta = \omega_1$ at the bottom, where $\theta = 0$, is obtained using the energy equation

$$\frac{1}{2}L_1^2\omega_1^2 = gL_1(1-\cos\alpha_1) \quad \text{or} \quad \omega_1^2 = \frac{4g}{L_1}\sin^2(\alpha_1/2).$$

At the bottom the length changes instantaneously from L_1 to $L_2 < L_1$, and since the velocity change is towards the point of support, angular momentum, $mL^2\dot\theta$, is conserved and immediately after this length change the angular velocity, ω_2, is larger and given by

$$L_1^2\omega_1 = L_2^2\omega_2 \quad \text{or} \quad \omega_2 = \frac{L_1^2}{L_2^2}\omega_1 > \omega_1. \tag{12.6}$$

The amplitude, α_2, of the next quarter swing can be related to ω_2 using the energy equation again,

$$\frac{1}{2}L_2^2\omega_2^2 - gL_2 = -gL_2\cos\alpha_2 \quad \text{or} \quad \omega_2^2 = \frac{4g}{L_2}\sin^2\left(\frac{\alpha_2}{2}\right).$$

But since ω_1 and ω_2 are related by equation 12.6 we can obtain a relation between successive amplitudes and the lengths L_1 and L_2,

$$L_1^3\sin^2\left(\frac{\alpha_1}{2}\right) = L_2^3\sin^2\left(\frac{\alpha_2}{2}\right). \tag{12.7}$$

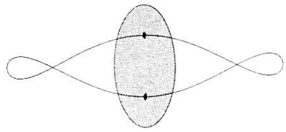
When the swing reaches its maximum amplitude, $\theta = \alpha_2$, its length is returned instantaneously to L_1. Again angular momentum is conserved, but at this point on the swing the angular velocity is zero so remains unchanged. Thus the swing starts its next half cycle with length L_1 but from the larger amplitude α_2.

This procedure can be performed on each swing, so after passing through the lowest point N times we have, on setting $L_1 = L_2 + h$,

$$\sin\left(\frac{\alpha_{N+1}}{2}\right) = \left(1 + \frac{h}{L_2}\right)^{3N/2}\sin\left(\frac{\alpha_1}{2}\right) \simeq \exp\left(\frac{3Nh}{2L_2}\right)\sin\left(\frac{\alpha_1}{2}\right).$$

Thus the amplitude increases *exponentially* with N. After a finite number of swings the right hand side of this equation becomes larger than unity and the theory is invalid: this happens when the system has sufficient energy for the swing to rotate round the support. In practice a real swing does not increase its amplitude so rapidly because the changes in length take place more gradually and not precisely at the optimum point.

Another simple example of parametric resonance is the child's toy, shown in the diagram, comprising a disc suspended on a loop of thread which can be made to spin alternately in opposite directions by pulling the loops twice in each complete cycle.



In both these examples large amplitude motion is produced by changing the system parameters with a frequency of *twice* the natural frequency of the system. This is typical, but the frequency does not have to be exactly twice the natural frequency, ω_0: the system can be parametrically pumped with a nearby frequency, and one important problem is to determine the width of the frequency band round $2\omega_0$ in which resonances occur. Parametric pumping can also occur near the frequencies $2\omega_0/n$, for integer n; in the example of the swing this is fairly obvious as we need only pump on every alternate pass through the bottom, for instance.

12.3.3 O Botafumeiro: parametric pumping in the middle ages

One of the oldest recorded examples of parametric pumping is swinging the giant censer, O Botafumeiro, in the cathedral of Santiago in Santiago de Compostela, a town in Galicia in northwest Spain. This cathedral was a pilgrims' shrine, famous throughout Christendom during the middle ages. The censer, with coals, weighs about 57 kg and hangs in the transept; it swings on a rope about 21m long with a maximum amplitude of about 80° and has a period of about 10 seconds; near the bottom of the swing it is travelling at about 68 km/hr (about 40 mph) half a metre above the floor.

Producing and maintaining such motion is far from trivial, but the rite of pumping O Botafumeiro appears to be about 700 years old, for the cathedral was built between the years 1078 to 1211, on the site of an older one destroyed by Almanzor, the military commander of the Moorish caliphate of Córdoba, in AD 997. The first recorded use of the censer is a 14th century margin note in the Codex, Liber Sancti Jacobi donated to the cathedral in about 1150: so we know that the rite of pumping O Botafumeiro started between 1150 and 1325, that is at least four centuries before the pendulum was studied scientifically.

The motion is started by moving the censer off the vertical to let it swing like a pendulum; then a team of men cyclically pull at cords attached to the upper end of the rope in order to decrease and increase its length as it passes through the lowest and highest points of the motion, as shown schematically in figure 12.1. Here the similarity with the child's swing ends. In that example of parametric pumping no formal rules are needed; a child never formulates the mechanism by which it pumps the swing. But O Botafumeiro needs a team effort, the chief verger calling orders where required; to obtain motion with an amplitude of about 80° roughly 17 pumping cycles are required and the total time taken is about 80 seconds. There is some evidence that at some point the rules for pumping became understood and transmitted to other local cathedrals; there are records of the cathedrals at Orense and Tuy (respectively 100 km SE and S of Santiago), but there are no records of the large gold censer at old St Peter's in Rome ever having been set in motion.

The motion of O Botafumeiro is said to be an impressive sight. It clearly imposes significant strains on the supports, with the result that several accidents have occurred. When pumping, the highest tensions in the ropes occur at the top and bottom of the swing, so it is here that the rope is most likely to break; the recorded accidents support this view. In 1622 the rope broke and the censer fell vertically, just missing the men pulling at the rope, suggesting that the break occurred near the highest point. In 1499 the chains attached to the censer broke and it landed at the side of the transept, crushing the door about 30 m from the centre of the swing; this could happen only if the amplitude of the motion was large and the break occurred near the bottom of the swing.

The dynamics of O Botafumeiro is complicated; in particular the amplitude of the motion is too large for the linear approximation to be made, the rope is heavy so its mass needs to be taken into account, partly because at the highest point of the swing it is far from straight, and air resistance needs to be included. Nevertheless this system *dle ages*

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has been studied, Sanmartín (1984), and the theoretical motion agrees well with its actual motion.

12.3.4 *One-dimensional linear systems*

First-order linear systems are relatively easy to understand and serve as a useful stepping stone to more complicated systems. The most general homogeneous, first-order system has the form

$$\frac{dx}{dt} = a(t)x, \tag{12.8}$$

where $a(t)$ is a T-periodic function of time, $a(t+T) = a(t)$ for all t. This equation can be solved by direct integration,

$$x(t) = x(t_0)\exp\left(\int_{t_0}^{t} ds\, a(s)\right), \tag{12.9}$$

but it is more helpful to use methods that can easily be generalised to higher-dimensional systems that cannot be integrated.

The linearity of the equations and the periodicity of the coefficient impose constraints upon the possible types of solution: if $x(t)$ is a solution then the function $y(t) = x(t+T)$ is also a solution because

$$\begin{aligned}\frac{dy}{dt} = \frac{d}{dt}x(t+T) = \frac{dx(t+T)}{d(t+T)} &= a(t+T)x(t+T)\\ &= a(t)y(t).\end{aligned}$$

Hence $x(t+T)$ and $x(t)$ satisfy the same equation. They are, however, *not* necessarily the same function, but because the differential equation is first-order and linear there is only one linearly independent solution, so we must have

$$x(t+T) = cx(t), \tag{12.10}$$

where c is a constant, independent of t.

Exercise 12.13

Show that for any integer n,

$$x(t_0 + nT) = c^n x(t_0) \tag{12.11}$$

and use equation 12.9 to deduce that

$$c = \exp\left(\int_0^T dt\, a(t)\right).$$

What property of $a(t)$ is required for $x(t)$ to be periodic?

Equation 12.11 has a particularly simple interpretation. If $c > 1$ then c^n grows exponentially with increasing n, so the solution $x(t)$ also increases exponentially. But if $c < 1$ then $c^n \to 0$ as $n \to \infty$, so $x(t) \to 0$ as $t \to \infty$. In the special case $c = 1$, $x(t+T) = x(t)$, for all t, and the solution is periodic: this occurs only when the mean value of $a(t)$ is zero.