Part of Lecture 2
8-29-2012
ϕ501

Euler-Lagrange Equation

$$\mathcal{F}[y(x)] = \int_a^b dx\, f(y(x), y'(x), x)$$

where $y'(x) = \frac{dy}{dx}$

and f is smooth and differentiable

Suppose $y(x) \to Y(x)$ makes $\mathcal{F}[y]$ min or max.*

Then $\mathcal{F}[Y + \epsilon\eta(x)]$ should vanish to first order in ϵ.

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* — added stipulation — we only look at paths where $y(a) = y_a$; $y(b) = y_b$ for fixed y_a, y_b.

So $\eta(a) = \eta(b) = 0$

$$\mathcal{F}[Y + \epsilon\eta] = \mathcal{F}[Y] + \int_a^b dx \left[\frac{\partial f}{\partial y(x)} \cdot \epsilon\eta(x) + \frac{\partial f}{\partial y'(x)} \cdot \epsilon\eta'(x)\right]$$

$$= \mathcal{F}[Y]$$

$$0 = \int_a^b dx \left[\frac{\partial f}{\partial y}\eta + \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\eta\right) - \left(\frac{d}{dx}\frac{\partial f}{\partial y'}\right)\eta\right]$$

The last two terms are just a rewrite of $\frac{\partial f}{\partial y'}\eta'$

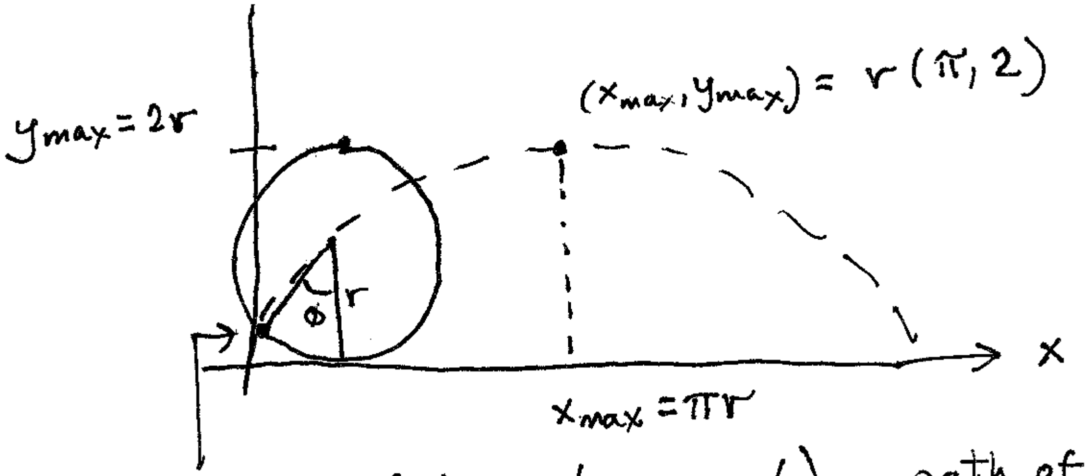
The middle term is $\left.\frac{\partial f}{\partial y'}\eta\right]_{x=a}^{x=b} = 0$

Since $\eta(x)$ is arbitrary (apart from end-point values)

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

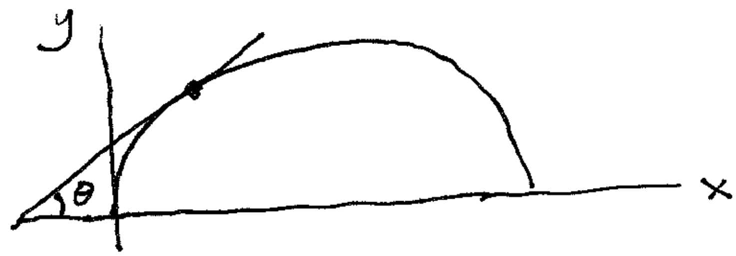
is required when $y \to Y(x) \equiv$ minimizing path.

Cycloid



$$(x,y) = r(\phi - \sin\phi,\ 1-\cos\phi)$$ path of cycloid.

$$(dx, dy) = r\,d\phi\,(1-\cos\phi,\ \sin\phi)$$



$$\frac{dy}{dx} = \tan\theta = \frac{\sin\phi}{1-\cos\phi}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \tan^2\theta = \frac{1}{\cos^2\theta} = \frac{1-\cos^2\phi}{(1-\cos\phi)^2} + 1 = \frac{2}{1-\cos\phi} = \frac{2r}{y}$$

$$\frac{v^2}{4gr} = \left(\frac{y}{2r} = \cos^2\theta\right)$$ — a nice geometric formula for the cycloid.

↑ from Newton: $v^2 = 2gy$ after falling a vertical distance y