Condition for amplification in parametric resonance

(Dated: October 17, 2012)

Here are some details that are missing from Landau & Lifshitz pp. 82-83. These details provide answers to questions raised today in class.

The equation of motion we want to solve is

$$d^2x(t)/dt^2 + \omega_0^2\left[1 + h\cos(2r)\right]x(t) = 0 \qquad (1)$$

where h is a small parameter measuring the strength of the parametric driving, and $r = (\omega_0 + \epsilon/2)t$ gives a driving frequency within ϵ of twice the resonant frequency $2\omega_0$ of the oscillator. So the other small parameter is $\epsilon/2\omega_0$.

The $2\omega_0$ value of the driving frequency will excite a certain amount of third harmonic response, varying as $c\cos(3r) + d\sin(3r)$. We therefore expect solutions of the type

$$x(t) = a(t)\cos(r) + b(t)\sin(r) + c(t)\cos(3r) + d(t)\sin(3r) + \cdots \qquad (2)$$

where probably the coefficients c, d will be small compared with a, b. We also expect the coefficients $a(t)$, $\cdots$ to vary slowly in time. In other words, the ratio $(da/dt)/a$ is small, of order ϵ/ω_0 or h. The second time derivative, $(d^2a/dt^2)/a$, should be of second order in smallness, and is therefore neglected. Our aim is to find out whether $a(t)$ or $b(t)$ could be growing in time as $\exp(st)$ with positive s. This behavior characterizes parametric resonance. Therefore, we look for solutions $a(t) = a\exp(st)$, $b(t) = b\exp(st), \cdots$. Slow variation of $a(t)$ means $|s|/\omega_0$ is also a small parameter. A pure imaginary exponent s would indicate a shifted frequency, and a complex s would have a real part indicating amplification.

To first order in small parameters, we have

$$d^2x/dt^2 + \omega_0^2 x = [(-\epsilon a + 2sb)\cos(r) + (-\epsilon b - 2sa)\sin(r) - ((8\omega_0 - 9\epsilon)c + 6sd)\cos(3r) + \cdots]\omega_0 e^{st} \qquad (3)$$

In the driving terms, we use trigonometric identities $\cos(mr)\cos(nr) = [\cos(m+n)r + \cos(m-n)r]/2$ and $\cos(mr)\sin(nr) = [\sin(m+n)r - \sin(m-n)r]/2$ to write

$$\omega_0^2 h\cos(2r)x = [(a+c)\cos(r) - (b+d)\sin(r) + a\cos(3r) + \cdots] \times \omega_0^2 h e^{st}/2 \qquad (4)$$

Adding Eqs.(3) and Eq.(4), and setting to zero the coefficient of $\cos(r)$, $\sin(r)$, $\cos(3r), \cdots$, we get the linear equation system

$$(-\epsilon + \omega_0 h/2)a + 2sb + (\omega_0 h/2)c = 0 \qquad (5)$$

$$2sa + (\epsilon + \omega_0 h/2)b + (\omega_0 h/2)d = 0 \qquad (6)$$

$$(\omega_0 h/2)a - (8\omega_0 - 9\epsilon)c - 6sd = 0 \qquad (7)$$

Eq.(7) (zeroing the coefficient of $\cos(3r)$) tells us that the amplitude $c = (h/16)a$ of the third harmonic cosine response is indeed a small number. The same is true for d, the amplitude of the third harmonic sine response. Therefore in Eqs.(5) and (6), the terms involving c and d should be neglected to lowest order. Thus we have the system of equations

$$\begin{pmatrix} -\epsilon + \omega_0 h/2 & 2s \\ 2s & \epsilon + \omega_0 h/2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0. \qquad (8)$$

These equations have solutions if $s = \pm\sqrt{(\omega_0 h/2)^2 - \epsilon^2}$. There is amplification (s real and positive) only if the driving frequency ω agrees with twice the resonance frequency $2\omega_0$ within $|\Delta\omega| = |\epsilon| < \omega_0 h/2$. Our assumptions are confirmed that $|\epsilon|/\omega_0$ and $|s|/\omega_0$ are both small when h is small, in the regime of amplification.

How do we know that the solution involves amplification ($s > 0$) rather than damping ($s < 0$)? The initial conditions always (except for one isolated point) imply that an amplifying term exists. The coefficients a, b must satisfy Eq.(8). This fixes the ratio $(b/a)_\pm = \pm(\epsilon - \omega_0 h/2)/2|s|$. The general solution is a sum of amplifying and decaying parts,

$$x(t) = A(\cos(r) + (b/a)_+\sin(r))e^{|s|t} + B(\cos(r) + (b/a)_-\sin(r))e^{-|s|t}. \qquad (9)$$

The coefficients A, B are set by initial conditions.