# Free Rotation of a symmetric top

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The behavior is derived both by geometric and by algebraic arguments.


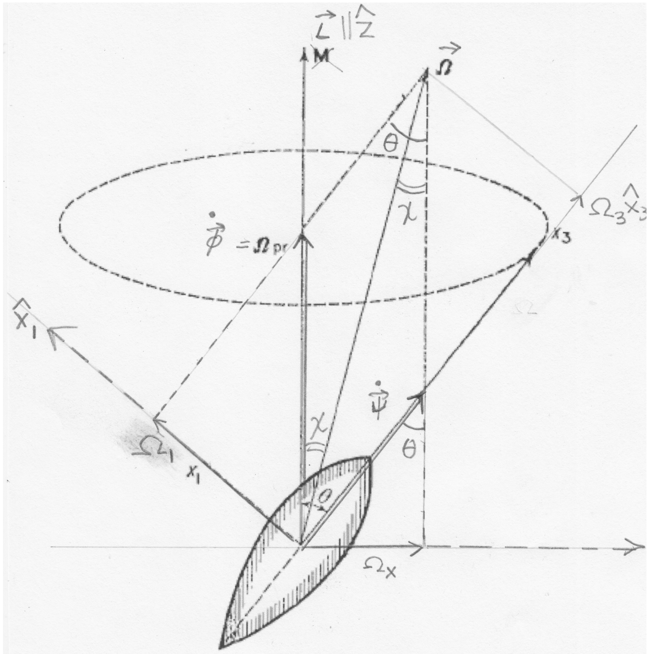


FIG. 1. This is figure 46, p107 from Landau and Lifshitz.

## I. INTRODUCTION

The symmetric top has inertia $I_3$ around a symmetry axis, and $I_1$ around the two perpendicular principle axes of rotation. In the body frame, the direction of the symmetry axis is $\hat{x}_3$. The directions of the other two axes, $\hat{x}_1$ and $\hat{x}_2$ can be chosen freely around the $\hat{x}_3$ axis. Total angular momentum $\vec{L}$ is constant. Choose the $\hat{Z}$ axis of the lab (or fixed, or inertial, or space) frame to align with $\vec{L}$. The system is shown in the figure, taken from Landau and Lifshitz, with some symbols added.

This picture shows the system in the lab frame, where Euler angle $\theta$ gives the tilt, and Euler angle $\phi$, at the instant shown, is zero. There is no attempt to show the angle $\psi$ – no special point on the body has been chosen. The body is symmetric around its axis $\hat{x}_3$. Notice that the angular velocity $\vec{\Omega}$ is shown, deviating from the angular momentum $\vec{L}$ by an angle labelled as $\chi$. The vectors $\hat{Z}$ and $\hat{x}_3$ ($z$-axes of the two frames), deviate from each other by $\theta$. $\vec{\Omega}$ is time-dependent in both lab and rotating frames. Can we say anything simple about this time dependence?

The important thing about this diagram is that the angular velocity $\vec{\Omega}$ has different decompositions, each uniquely determined. In the body frame (also called the rotating frame) it is $\vec{\Omega} = \Omega_1\hat{x}_1 + \Omega_2\hat{x}_2 + \Omega_3\hat{x}_3$. In the lab frame it is $\vec{\Omega} = \Omega_X\hat{X} + \Omega_Y\hat{Y} + \Omega_Z\hat{Z}$. In Eulerian angles, it is $\vec{\Omega} = \dot{\vec{\phi}} + \dot{\vec{\psi}} + \dot{\vec{\theta}}$. By definition of the Euler angles, $\dot{\vec{\phi}}$ lies along $\hat{Z}$, $\dot{\vec{\psi}}$ along $\hat{x}_3$, and $\dot{\vec{\theta}}$ along the "line of nodes," $\hat{Z}\times\hat{x}_3$. We can choose coordinates so that at time $t = 0$, the angular momentum $\vec{L} = L_1\hat{x}_1 + L_3\hat{x}_3$ is in the $(\hat{x}_1, \hat{x}_3)$-plane of the rotating system. Since $L_1 = I_1\Omega_1$, etc., we see that $\Omega_2 = 0$, and thus that the angular velocity $\vec{\Omega} = \Omega_1\hat{x}_1 + \Omega_3\hat{x}_3$ has also no $\hat{x}_2$ component, and is coplanar with $\vec{L} \parallel \hat{Z}$ and $\hat{x}_3$. Not only is $\Omega_2 = 0$, but also in the lab frame, we choose $\hat{X}$ and $\hat{Y}$ so that $\Omega_Y = 0$. In the Eulerian system, $\dot{\vec{\theta}} = 0$ because otherwise it would have a non-zero component $\Omega_Y$ along $\hat{Y}$ and $\Omega_2$ along $\hat{x}_2$.

As time evolves, the inertia axes move relative to $\vec{L}$ and $\vec{\Omega}$, and the components $\Omega_2$ and $\Omega_Y$ become non-zero. But this is not true for the $\theta$-component. The symmetry axis $\hat{x}_3$ of the top evolves in a cone around $\vec{L}$ as shown in the figure. To see this, consider a point $\vec{r}$ that is fixed in the body, and that happens to be on the $\hat{x}_3$ axis. The fundamental connection is that the velocity in the inertial frame $\vec{v} = d\vec{r}/dt$ of such a point is $\vec{\Omega}\times\vec{r}$, showing that the velocity is perpendicular to both $\vec{\Omega}$ and $\hat{x}_3$. As the two vectors $\vec{\Omega}$ and $\hat{x}_3$ evolve in time in the inertial frame, $\vec{v}$ continues to be perpendicular to both, and has constant magnitude. This shows that the component $\dot{\vec{\theta}}$ of angular velocity stays always zero. In general (for an unsymmetric top, or a symmetric top under gravitational torque) $\dot{\vec{\theta}}$ would not be zero. If at the instant ($t = 0$) of the figure, gravity parallel to $\hat{Z}$ were turned on, gravitational torque would give $\vec{L}$ a component in the direction of the line of nodes. The top would begin to nutate, and $\dot{\vec{\theta}}$ would be suddenly non-zero. But with no external torque, there is nothing to increase $\dot{\vec{\theta}}$ from zero.

So far we found one simple statement about the time-dependence of the free motion of the symmetric top: It moves with the tilt angle $\theta$ constant ($\dot{\theta} = 0$.) There are closely related statements that also follow. The angular momentum component $\Omega_3$ of the rotation around the symmetry axis is constant, and the magnitude of the part of $\vec{\Omega}$ perpendicular to $\hat{x}_3$ is constant. Equivalently, $\Omega_2^2 + \Omega_3^2 = \Omega_\perp^2$ is constant. And both $\dot{\phi}$ and $\dot{\psi}$ are constants. These are all evidently true once we accept that all the vectors shown in the figure simply rotate (in the inertial frame) around the fixed vector $\vec{L}$.

## II. EQUATIONS OF MOTION

The starting point is the equation for kinetic energy., $T = \frac{1}{2}\sum_{\alpha\beta} I_{\alpha\beta}\Omega_\alpha\Omega_\beta$. This equation is awkward to use in the lab frame, where the angular momentum components are $(\Omega_X, \Omega_Y, \Omega_Z)$, because the components of the tensor, like $I_{XY}$, vary in time as the top rotates, and need to be expressed in terms of the simpler components of $\mathbf{I}$ in the body frame. The kinetic energy in the body frame is $(I_1\dot{\Omega}_1^2 + I_2\dot{\Omega}_2^2 + I_3\dot{\Omega}_3^2)/2$. Unfortunately, we cannot use the simple Lagrange-type equation of motion $d(\partial\mathcal{L}/\partial\dot{\Omega}_i)/dt = \partial\mathcal{L}/\partial\Omega_i = 0$ because the angles $\Omega_1, \Omega_2, \Omega_3$ are not proper coordinates. There are two options.

(A) Re-write $\dot{\Omega}_i$ in terms of $\dot{\phi}, \dot{\theta}, \dot{\psi}, \phi, \theta, \psi$. Then equations like $d(\partial\mathcal{L}/\partial\dot{\phi}_i)/dt = \partial\mathcal{L}/\partial\phi$ are well-defined, but not especially simple. The Lagrangian is then, for the symmetric top (Landau & Lifshitz, eq. 35.2)

$$\mathcal{L} = T = \frac{1}{2}I_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \frac{1}{2}I_3(\dot{\phi}\cos\theta + \dot{\psi})^2 \quad (1)$$

From this one can show that $\dot{\phi}$, $\dot{\theta}$, and $\dot{\psi}$ are all constant, and by an argument simplified by judicious choice of initial conditions, $\dot{\theta} = 0$ (Landau & Lifshitz, eq. 35.4.)

(B) Use the inertial-frame Newtonian version $d\vec{L}/dt = \vec{\tau} = 0$ (where $\vec{\tau}$ is the torque, zero for free rotation), and transform to the body frame where $d\vec{L}/dt = -\vec{\Omega} \times \vec{L}$. In this frame we can use $L_i = I_i\Omega_i$. This procedure yield the Euler equations (Landau & Lifshitz, eq. 36.4, specialized to the symmetric case $I_1 = I_2$.)

$$\begin{aligned} d\Omega_1/dt &= -[\Omega_3(I_1 - I_3)/I_1]\Omega_2 \\ d\Omega_2/dt &= +[\Omega_3(I_1 - I_3)/I_1]\Omega_1 \\ d\Omega_3/dt &= 0 \end{aligned} \quad (2)$$

From the third of these we have that $\Omega_3$ is constant, as deduced above. Because of this, the factor in square brackets in the first two equations is a constant frequency. Solving the equations is easy, and yields a uniform rotation of the perpendicular components of $\vec{\Omega}$ around the axis $\hat{x}_3$ of the top, as seen in the body frame. This tells us that $\dot{\theta}$=0, or the tilt angle $\theta$ is a constant. All this was also deduced above from the diagram. The frequency of this rotation is in fact just $\dot{\psi} = [\Omega_3(I_1 - I_3)/I_1]$.

Interpreting this requires thought. If you are in the body (rotating) frame, the body is not rotating, but the inertial frame (the stars if you are standing in place on earth) are rotating. If $\dot{\psi}$ is small compared with $\Omega_3$ (as is the case on earth where the factor $(I_1 - I_3)/I_1 \approx 0.0033$) then what you see is the rotation of the stars around the $\hat{x}_3$ axis of symmetry, plus a much slower precession of the point in the sky (Polaris for now) we rotate around. The axis of rotation, as located this way in the sky, evolves in a circle at angular velocity $\dot{\psi}$. Earth's precession from this effect is quite small, and perturbed by elasticity. A much slower precession (period 26,000 years) with larger amplitude, is caused by torques from the sun and the moon.

From these equations we learned one thing that wasn't yet learned from the diagram, namely that $\dot{\psi} = \Omega_3(1 - I_3/I_1)$. This is in fact, an interesting (and surprising!) result, so we will next learn how to deduce it from the diagram. Why is it surprising? Because the precession frequency does not depend on the angle of inclination ($\chi$, for the inclination of $\vec{\Omega}$ relative to the symmetry axis $\hat{x}_3$, or $\theta$ for the inclination of the angular momentum to the symmetry axis.) It is a harmonic-oscillator type motion, where the frequency is independent of amplitude.

## III. PROPERTIES OF $\vec{\dot{\phi}}$ AND $\vec{\dot{\psi}}$ DEDUCED FROM THE DIAGRAM

We have $\dot{\theta} = 0$ and $\vec{\Omega} = \vec{\dot{\phi}} + \vec{\dot{\psi}}$. We want to find formulas for the components $\dot{\phi}$ and $\dot{\psi}$ of $\vec{\Omega}$ on the non-orthogonal axes $\hat{Z}$ and $\hat{x}_3$. First, project onto axes perpendicular to $\hat{x}_3$ (this is $\hat{x}_1$; the projection is $\Omega_1$) and perpendicular to $\hat{Z}$ (this is $\hat{X}$; the projection is $\Omega_X$). These projections are both shown on the diagram. Then the desired component times $\sin\theta$ is just this projection, so we have

$$\begin{aligned} \dot{\phi}\sin\theta &= \Omega_1 \\ \dot{\psi}\sin\theta &= \Omega_X \end{aligned} \quad (3)$$

Landau and Lifshitz simplify the first of these by noticing that $\Omega_1 = L_1/I_1 = L\sin\theta/I_1$, which gives $\dot{\phi} = M/I_1$. They call this frequency $\Omega_{\rm pr}$, because it is the rate of precession of the symmetry axis of the top, seen in the lab frame.

To simplify the second equation, notice that $\Omega_X = \Omega\sin\chi$, so $\dot{\psi} = \Omega\sin\chi/\sin\theta$. Next, write $\sin\chi = \sin[\theta - (\theta - \chi)]$, and notice that $\theta - \chi$ is the angle between $\vec{\Omega}$ and $\hat{x}_3$. Therefore $\sin(\theta - \chi) = \Omega_1/\Omega$ and $\cos(\theta - \chi) = \Omega_3/\Omega$. This allows us to write $\dot{\psi} = \Omega[\Omega_3/\Omega - (\Omega_1/\Omega)(\cos\theta/\sin\theta)]$. Finally, $\cos\theta/\sin\theta = L_3/L_1 = I_3\Omega_3/I_1\Omega_1$. This gives us the result $\dot{\psi} = \Omega_3(1 - I_3/I_1)$. The surprising result from the Euler equations is thus also already contained in the diagram.

To me, the very counter-intuitive part is the contrast between the precession $\dot{\phi}$ seen in the lab frame, and the precession $\dot{\psi}$ seen in the body frame. This contrast is very large for an object like earth with $I_3 \approx I_1$. The lab frame say the object rotates around its symmetry axis at the slow rate $\dot{\psi}$, whereas the body frame says the rate of rotation around the symmetry axis, perceived by watching the sky, is $\Omega_3$. According to the decomposition of the Euler angular velocities, $\Omega_3 = \dot{\psi} + \dot{\phi}\cos\theta$ (see eq.(1), very different from $\dot{\psi}$, and dominated by the other part, $\dot{\phi}\cos\theta$. The confusing thing is, why does the observer on the sun interpret so much of this ($\dot{\phi}$) as precession of the body axis, and so little as rotation around the body axis ($\dot{\psi}$)? Since the angle $\theta - \chi$ for the earth is very small, the two types of rotation must be very hard to distinguish in the lab frame, which makes it a little less counter-intuitive.