Physics 501 Fall 2012 Class 3 Aug. 31, 2012

Functionals $\mathcal{F}[\phi]$ and functional derivatives $\dfrac{\delta \mathcal{F}}{\delta \phi(x)}$

An example of a functional is

$$\mathcal{F}[\phi] = \int_0^a dx' \, \frac{\phi^2(x') + a^2 (d\phi(x')/dx')^2}{a^2 + x'^2} = \int_0^a dx' \, f(\phi, \phi', x')$$

If you are given a function $\phi(x)$, you can compute the number $\mathcal{F}[\phi]$. The variation in the number $\mathcal{F}[\phi]$ when $\phi(x')$ is varied in the vicinity of $x' = x$, is related to the functional derivative $\delta\mathcal{F}/\delta\phi$.

Specifically, the definition is, that if $\phi(x')$ is changed to $\phi(x') + \epsilon\eta(x')$ where $\epsilon \ll 1$, $\eta(x')$ is well-behaved, then

$$\mathcal{F}[\phi] \to \mathcal{F}[\phi + \epsilon\eta] \cong \mathcal{F}[\phi] + \epsilon \int dx' \, \frac{\delta \mathcal{F}}{\delta \phi(x')} \eta(x').$$

And notice that, if $\eta(x') = \delta(x' - x)$, then

$$\mathcal{F}[\phi + \epsilon\eta] \to \mathcal{F}[\phi + \epsilon\delta_x] = \mathcal{F}[\phi] + \epsilon \frac{\delta \mathcal{F}}{\delta \phi(x)}$$

Thus the functional derivative can be defined as

$$\frac{\delta \mathcal{F}}{\delta \phi(x)} = \lim_{\epsilon \to 0} \frac{\mathcal{F}[\phi + \epsilon\delta_x] - \mathcal{F}[\phi]}{\epsilon}$$

where δ_x means a Dirac delta function located at x.

It is helpful to use $\int dx' g(x') \delta(x - x') = g(x)$ to define $\delta(x - x')$. Then it follows that the derivative $\delta'(x - x') = \frac{d}{dx'} \delta(x - x')$ has the property

$$\int dx' g(x') \delta'(x - x') = -g'(x)$$

For the case $\mathcal{F}[\phi] = \int dx' f(\phi, \phi', x')$

The Euler-Lagrange equation is

$$\frac{\delta \mathcal{F}}{\delta \phi(x)} = \left[\frac{\partial f}{\partial \phi} - \frac{d}{dx} \frac{\partial f}{\partial \phi'} \right]_{\phi = \phi(x),\ \phi' = \frac{d\phi(x)}{dx}}$$

and the functional derivative is zero whenever $\phi(x)$ is a function that makes $\mathcal{F}[\phi]$ an extremum.