"Recitation" 11-06-12

Spherical Pendulum

LL p33
HF p183

$$\mathcal{L} = \tfrac{1}{2} m R^2 (\dot\theta^2 + \sin^2\theta\,\dot\phi^2) + mgR\cos\theta$$

$$\mathcal{L}' = \frac{\mathcal{L}}{mgR} = \frac{1}{2}\frac{R}{g}(\dot\theta^2 + \sin^2\theta\,\dot\phi^2) + \cos\theta$$ dimensionless Lagrangian

$\sqrt{\frac{R}{g}}$ = unit of time. $\tau = t/\sqrt{\frac{R}{g}}$

$$\mathcal{L}' = \tfrac{1}{2}(\dot\theta^2 + \sin^2\theta\,\dot\phi^2) + \cos\theta$$

EoM

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\phi} = 0 \;;\; \frac{\partial \mathcal{L}}{\partial \dot\phi} = \boxed{P_\phi = \sin^2\theta\,\dot\phi} = \text{const}$$

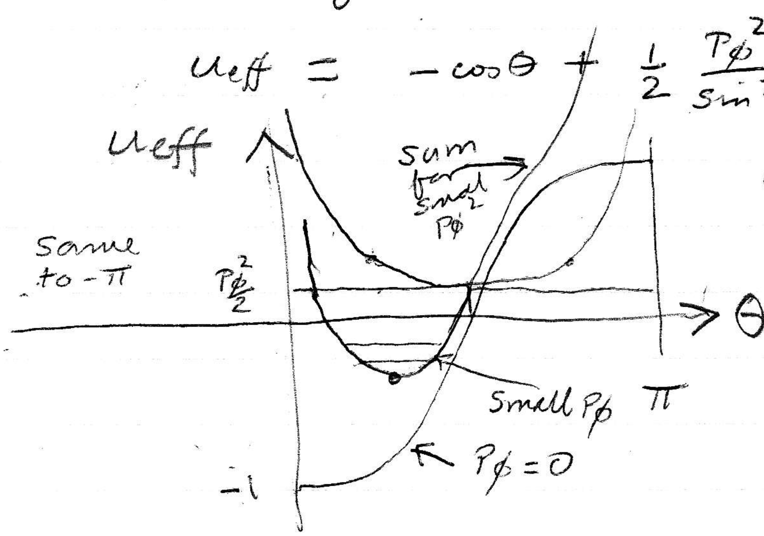
$$\ddot\theta = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\theta} = \frac{\partial \mathcal{L}}{\partial \theta} = \sin\theta\cos\theta\,\dot\phi^2 - \sin\theta$$

$$\boxed{\ddot\theta = P_\phi^2\frac{\cos\theta}{\sin^3\theta} - \sin\theta}$$

P_ϕ^2 is measured in units $\frac{mgR}{mR^2} = \frac{g}{R} = \omega_{pend}^2$

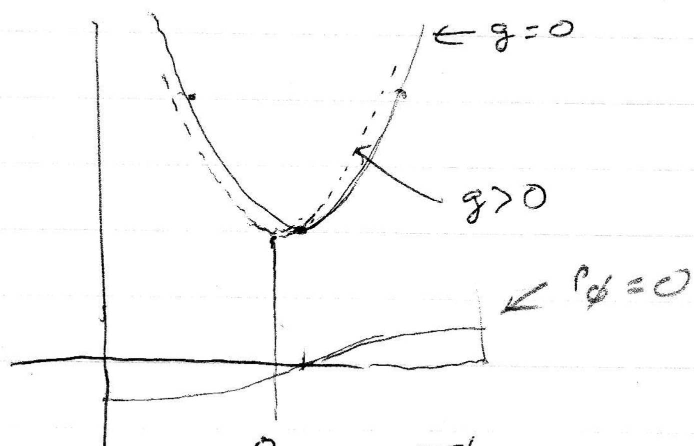
Energy $\frac{E}{mgR} = e = \frac{1}{2}\dot\theta^2 + u_{eff}(\theta)$

$$u_{eff} = -\cos\theta + \frac{1}{2}\frac{P_\phi^2}{\sin^2\theta}$$



θ-motion is periodic between θ_{min} + θ_{max}

Solve approximately for small θ-oscillations.
Simplest case – large P_ϕ



θ_0 near $\pi/2$

Let $\theta = \pi/2 + \epsilon$

$\sin\theta = \cos\epsilon$

$\cos\theta = -\sin\epsilon$

$$\frac{1}{2}\left(\frac{d\epsilon}{dt}\right)^2 + \frac{P_\phi^2}{2\cos^2\epsilon} + \sin\epsilon = e$$

expand for small ϵ $\qquad \cos^{-2}\epsilon \sim \left(1 - \epsilon^2/2\right)^{-2} \sim 1 + \epsilon^2$

$$U_{ef} = \frac{P_\phi^2}{2}(1+\epsilon^2) + \epsilon + \mathcal{O}(\epsilon^3)$$

$$= \frac{P_\phi^2}{2}\left(\epsilon + \frac{1}{P_\phi^2}\right)^2 + \left(\frac{P_\phi^2}{2} - \frac{1}{P_\phi^2}\right)$$

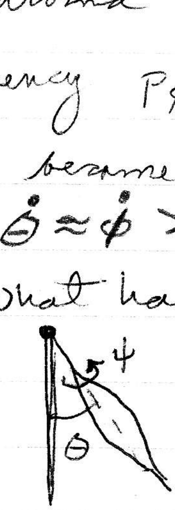
θ oscillates around $\theta_0 = \frac{\pi}{2} - \frac{1}{P_\phi^2}$

at frequency $P_\phi \gg 1$ $\quad$ ($\dot\theta, \dot\phi$ in units of $\sqrt{\frac{g}{\ell}} = \omega_{pend}$)

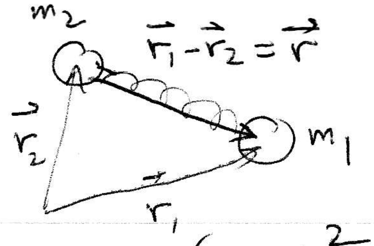
gravity became irrelevant. Motion in a tilted plane.

$\dot\theta \approx \dot\phi \gg \omega_{pendulum} = \sqrt{g/R}$

Question what happens if the pendulum can spin?



(But only around the axis of the massless support rod?)



$$\boxed{\text{2d : 2 particles + spring}}$$

$$U = \tfrac{1}{2}k\left(|\vec{r}_1 - \vec{r}_2| - d\right)^2 = \tfrac{1}{2}k(r-d)^2$$

$$T = \tfrac{1}{2}M\dot{\vec{R}}^2 + \tfrac{1}{2}\mu\dot{\vec{r}}^2$$

Free motion of CM. $\vec{R} = \vec{R}_0 + \vec{V}_0 t$

move to CM frame

$$\boxed{\mathcal{L} = \tfrac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \tfrac{1}{2}k(r-d)^2}$$

$$\frac{\partial\mathcal{L}}{\partial\theta} = 0 \Rightarrow \mu r^2\dot{\theta} = L_\theta = \text{const}$$

New units
$$r \to r/d$$
$$t \to \sqrt{\frac{k}{\mu}}\,t$$
$$\mathcal{L} \to \frac{\mathcal{L}}{kd^2}$$

$$\mathcal{L} = \tfrac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \tfrac{1}{2}(r-1)^2$$

$$\boxed{\begin{aligned} L_\theta &= r^2\dot{\theta} \\ E &= \tfrac{1}{2}\dot{r}^2 + U_{eff}(r) \\ U_{eff} &= \frac{L_\theta^2}{2r^2} + \tfrac{1}{2}(r-1)^2 \end{aligned}}$$

Low energy behavior : $U_{eff} \sim \frac{kT}{kd^2} \ll 1$

Requires $r = 1 + \epsilon$, $\epsilon \ll 1$

$$u_{eff} = \frac{L_\theta^2}{2(1+\epsilon)^2} + \frac{1}{2}\epsilon^2 \qquad \text{with } L_\theta \sim \mathcal{O}(\epsilon), \quad \epsilon \ll 1$$

$$\sim \frac{L_\theta^2}{2}\left(1 - 2\epsilon + 3\epsilon^2\right) + \frac{1}{2}\epsilon^2$$

$$\sim \frac{1}{2}(3L_\theta^2+1)\left(\epsilon - \frac{L_\theta^2}{(3L_\theta^2+1)}\right)^2 + \text{const}$$

spring constant shifts from 1 to $1+3L_\theta^2$

equilibrium position shift from 1 to $1+\frac{L_\theta^2}{1+3L_\theta^2}$

$$\omega_{sp} = \sqrt{\frac{k}{\mu}\left(1+3L_\theta^2\right)}$$

$$L_\theta = \left(1 + \frac{L_\theta^2}{1+3L_\theta^2}\right)\dot\theta \sim \left(1 + \frac{\dot\theta^2}{1+3\dot\theta^2}\right)\dot\theta$$

$$\sim \left(1+\dot\theta^2\right)\dot\theta$$