J. D. Jackson, Classical Electrodynamics, 2nd Ed. (Wiley, NY, 1975)

The conservation of linear momentum can be similarly considered. The total electromagnetic force on a charged particle is

$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \tag{6.113}$$

If the sum of all the momenta of all the particles in the volume V is denoted by $\mathbf{P}_{\text{mech}}$, we can write, from Newton's second law,

$$\frac{d\mathbf{P}_{\text{mech}}}{dt} = \int_V \left(\rho\mathbf{E} + \frac{1}{c}\mathbf{J} \times \mathbf{B}\right) d^3x \tag{6.114}$$

where we have converted the sum over particles to an integral over charge and current densities for convenience in manipulation. In the same manner as for Poynting's theorem, we use the Maxwell equations to eliminate ρ and $\mathbf{J}$ from (6.114):

$$\rho = \frac{1}{4\pi}\nabla \cdot \mathbf{E}, \qquad \mathbf{J} = \frac{c}{4\pi}\left(\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\right) \tag{6.115}$$

With (6.115) substituted into (6.114) the integrand becomes

$$\rho\mathbf{E} + \frac{1}{c}\mathbf{J} \times \mathbf{B} = \frac{1}{4\pi}\left[\mathbf{E}(\nabla \cdot \mathbf{E}) + \frac{1}{c}\mathbf{B} \times \frac{\partial \mathbf{E}}{\partial t} - \mathbf{B} \times (\nabla \times \mathbf{B})\right]$$

Then writing

$$\mathbf{B} \times \frac{\partial \mathbf{E}}{\partial t} = -\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) + \mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t}$$

and adding $\mathbf{B}(\nabla \cdot \mathbf{B}) = 0$ to the square bracket, we obtain

$$\rho\mathbf{E} + \frac{1}{c}\mathbf{J} \times \mathbf{B} = \frac{1}{4\pi}[\mathbf{E}(\nabla \cdot \mathbf{E}) + \mathbf{B}(\nabla \cdot \mathbf{B}) - \mathbf{E} \times (\nabla \times \mathbf{E}) - \mathbf{B} \times (\nabla \times \mathbf{B})] - \frac{1}{4\pi c}\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B})$$

The rate of change of mechanical momentum (6.114) can now be written

$$\frac{d\mathbf{P}_{\text{mech}}}{dt} + \frac{d}{dt}\int_V \frac{1}{4\pi c}(\mathbf{E} \times \mathbf{B})\, d^3x$$
$$= \frac{1}{4\pi}\int_V [\mathbf{E}(\nabla \cdot \mathbf{E}) - \mathbf{E} \times (\nabla \times \mathbf{E}) + \mathbf{B}(\nabla \cdot \mathbf{B}) - \mathbf{B} \times (\nabla \times \mathbf{B})]\, d^3x \tag{6.116}$$

We may tentatively identify the volume integral on the left as the total electromagnetic momentum $\mathbf{P}_{\text{field}}$ in the volume V:

$$\mathbf{P}_{\text{field}} = \frac{1}{4\pi c}\int_V (\mathbf{E} \times \mathbf{B})\, d^3x \tag{6.117}$$

The integrand can be interpreted as a density of electromagnetic momentum. We note that this momentum density is proportional to the energy-flux density $\mathbf{S}$, with proportionality constant c^{-2}.

To complete the identification of the volume integral of

$$\mathbf{g}=\frac{1}{4\pi c}(\mathbf{E}\times\mathbf{B}) \tag{6.118}$$

as electromagnetic momentum, and to establish (6.116) as the conservation law for momentum, we must convert the volume integral on the right into a surface integral of the normal component of something which can be identified as momentum flow. Let the Cartesian coordinates be denoted by x_α, $\alpha=1,2,3$. The $\alpha=1$ component of the electric part of the integrand in (6.116) is given explicitly by

$$\begin{aligned}[\mathbf{E}(\nabla\cdot\mathbf{E})-\mathbf{E}\times(\nabla\times\mathbf{E})]_1 &= E_1\left(\frac{\partial E_1}{\partial x_1}+\frac{\partial E_2}{\partial x_2}+\frac{\partial E_3}{\partial x_3}\right)-E_2\left(\frac{\partial E_2}{\partial x_1}-\frac{\partial E_1}{\partial x_2}\right)+E_3\left(\frac{\partial E_1}{\partial x_3}-\frac{\partial E_3}{\partial x_1}\right)\\ &=\frac{\partial}{\partial x_1}(E_1^2)+\frac{\partial}{\partial x_2}(E_1E_2)+\frac{\partial}{\partial x_3}(E_1E_3)-\frac{1}{2}\frac{\partial}{\partial x_1}(E_1^2+E_2^2+E_3^2)\end{aligned}$$

This means that we can write the αth component as

$$[\mathbf{E}(\nabla\cdot\mathbf{E})-\mathbf{E}\times(\nabla\times\mathbf{E})]_\alpha=\sum_\beta\frac{\partial}{\partial x_\beta}(E_\alpha E_\beta-\tfrac{1}{2}\mathbf{E}\cdot\mathbf{E}\delta_{\alpha\beta}) \tag{6.119}$$

and have the form of a divergence of a second rank tensor on the right-hand side. With the definition of the *Maxwell stress tensor* $T_{\alpha\beta}$ as

$$T_{\alpha\beta}=\frac{1}{4\pi}[E_\alpha E_\beta+B_\alpha B_\beta-\tfrac{1}{2}(\mathbf{E}\cdot\mathbf{E}+\mathbf{B}\cdot\mathbf{B})\delta_{\alpha\beta}] \tag{6.120}$$

Eq. (6.116) can therefore be written in component form as

$$\frac{d}{dt}(\mathbf{P}_{\text{mech}}+\mathbf{P}_{\text{field}})_\alpha=\sum_\beta\int_V\frac{\partial}{\partial x_\beta}T_{\alpha\beta}\,d^3x \tag{6.121}$$

Application of the divergence theorem to the volume integral gives

$$\frac{d}{dt}(\mathbf{P}_{\text{mech}}+\mathbf{P}_{\text{field}})_\alpha=\oint_S\sum_\beta T_{\alpha\beta}n_\beta\,da \tag{6.122}$$

where $\mathbf{n}$ is the outward normal to the closed surface S. Evidently, if (6.122) represents a statement of conservation of momentum, $\sum_\beta T_{\alpha\beta}n_\beta$ is the αth component of the flow per unit area of momentum across the surface S into the volume V. In other words, it is the force per unit area transmitted across the surface S and acting on the combined system of particles and fields inside V. Equation (6.122) can therefore be used to calculate the forces acting on material objects in electromagnetic fields by enclosing the objects with a boundary surface S and adding up the total electromagnetic force according to the right-hand side of (6.122).

The conservation of angular momentum of the combined system of particles