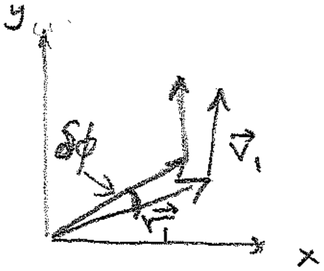
$\vec{L}_a = \vec{r}_a \times \vec{p}_a$ = angular momentum of particle a

$\mathcal{L} = \mathcal{L}(r_1, \cdots r_N, \dot{r}_1, \cdots, \dot{r}_N) = \sum_a \frac{1}{2} m_a v_a^2 - U(r_1 \cdots r_N)$

This is true for a closed system. $\dot{\vec{r}}_a = \vec{v}_a$

$$\delta\mathcal{L} = \mathcal{L}(\vec{r}_a + \delta\vec{\phi}\times\vec{r}_a,\ \vec{v}_a + \delta\phi\times\vec{v}_a) - \mathcal{L}(\vec{r}_a, \vec{v}_a)$$

= change of $\mathcal{L}$ under infinitesimal rotation.



$\delta\vec{\phi}$ = vector along axis of rotation.

$|\delta\vec{\phi}| = \delta\phi$ = infinitesimal angle in plane $\perp$ to $\delta\vec{\phi}$.

$\vec{r}_1 \to \vec{r}_1 + \delta\vec{r}_1 \qquad \delta\vec{r}_1 = \delta\vec{\phi}\times\vec{r}_1$

$\vec{v}_1 \to \vec{v}_1 + \delta\vec{v}_1 \qquad \delta\vec{v}_1 = \delta\vec{\phi}\times\vec{v}_1$

$\delta\mathcal{L}$ expanded to first order in $\delta\vec{\phi}$ leads to

$$\frac{\delta\mathcal{L}}{\delta\vec{\phi}} = \frac{d}{dt}\sum_a \vec{r}_a \times \frac{\partial\mathcal{L}}{\partial\vec{v}_a} = \frac{d}{dt}\vec{L}$$

where $\vec{L} = \sum_a \vec{r}_a \times \frac{\partial\mathcal{L}}{\partial\vec{v}_a} = \sum_a \vec{r}_a \times \vec{p}_a = \sum_a \vec{L}_a$

if $\frac{\delta\mathcal{L}}{\delta\vec{\phi}} = 0$ (as is true in a closed system)

then $\vec{L}$ = constant.

Note also that $\{\phi_a\}$ $(a = 1, \cdots, N)$ could be chosen as part of a set of generalized coordinates. ϕ_α is non-infinitesimal. It is in the plane $\perp$ to $\delta\vec{\phi}_a$

Then $\frac{\partial\mathcal{L}}{\partial\phi_a} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}_a}$

and $\frac{\partial\mathcal{L}}{\partial\dot{\phi}_a} = L_a$ and $\frac{\partial\mathcal{L}}{\partial\vec{\phi}} = \hat{z}\sum_a \frac{\partial\mathcal{L}}{\partial\phi_a} = L_z = \sum_a (\vec{L}_a)_z \hat{z}$

So $L_{az} = \frac{\partial\mathcal{L}}{\partial\dot{\phi}_\alpha}$ if ϕ_α is an angle of rotation of particle a around the $\delta\vec{\phi}$ or $\hat{z}$ axis.

Example

particle (mass m, charge q, field $\vec{B} = B\hat{z}$)

$\mathcal{L} = \frac{1}{2}mv^2 + q\vec{v}\cdot\vec{A}$ where $\vec{B} = \vec{\nabla}\times\vec{A}$

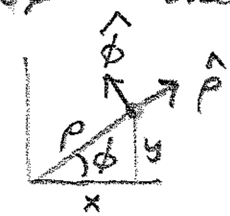
Uniform field: $\vec{A} = \frac{1}{2}(\vec{B}\times\vec{r}) = \frac{B}{2}(y\hat{x} - x\hat{y})$

(Note: I forgot the factor $\frac{1}{2}$ in class. Also note — this is the symmetric "Coulomb gauge". The "Landau gauge" is $\vec{A} = -Bx\hat{y}$)

cylindrical coordinates $\vec{r} = (\rho\cos\phi, \rho\sin\phi, z)$

$\vec{r} = \rho\hat{\rho} + z\hat{z}$. $\vec{v} = \dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi} + \dot{z}\hat{z}$

Note $\hat{\phi}$ is a unit vector in the xy plane, not related to $\delta\vec{\phi}$ — the infinitesimal rotation around $\hat{z}$.



$\hat{\rho} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$

$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$

$\hat{\rho}\cdot\hat{\phi} = 0$

$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left[\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right]$

$\frac{\partial T}{\partial\dot{\phi}} = L_z = m\rho^2\dot{\phi}$ is conserved since $\frac{\partial\mathcal{L}}{\partial\phi} = 0$ in a closed system.

In a constant $\vec{B}$-field, $\vec{A} = \frac{B}{2}\rho\hat{\phi}$

$\vec{v}\cdot\vec{A} = \frac{B}{2}\rho^2\dot{\phi}$

$$\mathcal{L} = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2 + \frac{1}{2}m\dot{z}^2 + \frac{qB}{2}\rho^2\dot{\phi}$$

Since $\frac{\partial\mathcal{L}}{\partial\phi} = 0$, $\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = m\rho^2\dot{\phi} + q\frac{B}{2}\rho^2$ is conserved.

In class, it was implicitly assumed that the origin was at the center of the circular orbit — i.e. that $\rho = r_c = \frac{mv_\perp}{qB}$. Then $\rho = $ const and $m\rho^2\dot{\phi} = L_z$ is conserved when $\partial B/\partial t = 0$.

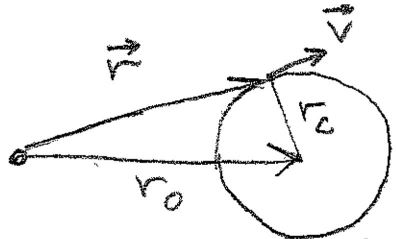
When $\partial B/\partial t \neq 0$, then it is still true that $\left[m\rho^2\dot{\phi} + \frac{qB}{2}\rho^2\right]$ is conserved, but neither part is separately conserved. Faraday induction effects cause $L_z = m\rho^2\dot{\phi}$ to change, compensating the change in $\frac{qB}{2}\rho^2$.

<u>But</u>, if the origin is not at the center of the orbit, then the equation is interesting even when $\partial B/\partial t = 0$.

$$r_c = \frac{m v_\perp}{qB}$$

$\vec{\rho} = \vec{r}_\perp$ (component of $\vec{r}$ in xy plane)

$\vec{\rho}_0 = \vec{r}_{0\perp}$ "



It is always true that $L_z = (\vec{r} \times m\vec{v})\cdot\hat{z} = \rho^2\dot{\phi}$
(That's just geometry.) But it is clearly <u>Not</u> conserved around most origins. So, what <u>is</u> conserved?

Look carefully at the two pieces of [ ] above.

$$m\rho^2\dot{\phi} = \frac{m}{B}\vec{B}\cdot(\vec{r}\times\vec{v}) = L_z \text{ from geometry}$$

$$= \frac{m}{B}\vec{v}\cdot(\vec{B}\times\vec{r}) \quad \text{(cyclic permutation)}$$

$$\frac{qB}{2}\rho^2 = \frac{q}{2B}(\vec{B}\times\vec{r})\cdot(\vec{B}\times\vec{r})$$

$$M_z \equiv \left[m\rho^2\dot{\phi} + \frac{1}{2}\frac{qB}{2}\rho^2\right] = \frac{1}{B}\left[m\vec{v} + \frac{q}{2}(\vec{B}\times\vec{r})\right]\cdot(\vec{B}\times\vec{r}) = \frac{1}{B}\vec{P}\cdot(\vec{B}\times\vec{r})$$

where $\vec{P} \equiv \frac{\partial \mathcal{L}}{\partial \vec{v}} = m\vec{v} + q\vec{A}$ is the new ("canonical") momentum

and $M_z = \hat{z}\cdot(\vec{r}\times\vec{P})$ by cyclic permutation (where $\hat{z} = \vec{B}/B$)

The canonical "angular momentum" is $\vec{r}\times(m\vec{v} + q\vec{A})$ and is conserved <u>for all orbits</u>, independent of origin. The two terms have compensating time-variation.