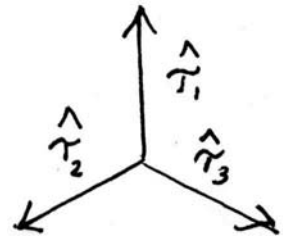
Phy 540 Fall 2005 Final Exam Tuesday Dec. 20

8:00-10:30 am room B-131

Closed book. Calculators allowed.

Do 5 out of 6 problems.



1. **A random walk** on the two-dimensional triangular lattice consists of a random path $\boldsymbol{R}$ which is a sum of N steps $\boldsymbol{r}_i$, $i=1,2,...,N$ where $\boldsymbol{r}_i$ is chosen randomly from the six vectors τ_1, τ_2, τ_3 (unit vectors at 120° as shown) plus τ_4, τ_5, τ_6 which are the negatives of τ_1, τ_2, τ_3. Find the mean square displacement $<|\boldsymbol{R}|^2>$.

2. **Bose Condensation.** The density of orbitals per unit energy, for non-relativistic, non-interacting particles of mass m in d spatial dimensions is $D(\varepsilon)=N(\pi^{d/2}/\Gamma(d/2))(\varepsilon/E_0)^{d/2-1}/E_0$ where $E_0=\hbar^2k_0^2/2m$, and $k_0=2\pi n^{1/d}$, where n is the d-dimensional particle density N/V.

a. Derive this answer in d=3 where $\Gamma(3/2)=\pi^{1/2}/2$.

b. Write a formula (leave it as a one-dimensional integral over the density of states $D(\varepsilon)$) for the number $N(\mu,T)$ of occupied states at chemical potential μ and temperature T for a non-interacting non-relativistic Bose gas, in arbitrary dimension, ignoring the possible phenomenon of Bose condensation. $N(\mu,T)$ will later be tuned (by adjustment of μ) to equal N, the actual particle number.

c. For what values of p is the integral

$$\int_0^\infty dx \frac{x^p}{(e^x-1)}$$

convergent, and for what values of p is it divergent?

d. For $d=1,2,3,4$, explain the occurrence or non-occurrence of Bose condensation.

3. **Silly model of ice:** There are N molecules in volume V which do not interact, so it doesn't matter much where they are. Each has 3 point charges, one with charge $q_0=-2e$, and two with $q_1=q_2=+e$. You can imagine a cube of side $2a$ where a=1.00 Å with the $q=-2e$ charge at the center, and the $q=+e$ charges randomly occupying corner sites. Four corner sites are allowed, $\boldsymbol{r}_0=(a/3^{1/2})(1,1,1)$, $\boldsymbol{r}_1=(a/3^{1/2})(1,-1,-1)$, $\boldsymbol{r}_2=(a/3^{1/2})(-1,1,-1)$, and $\boldsymbol{r}_3=(a/3^{1/2})(-1,-1,1)$. These lie on the corners of a tetrahedron. The other four corner sites are forbidden. Two charges cannot be on the same corner site. In the absence of an external electric field, the various allowed occupancies of corner sites all have the same energy.

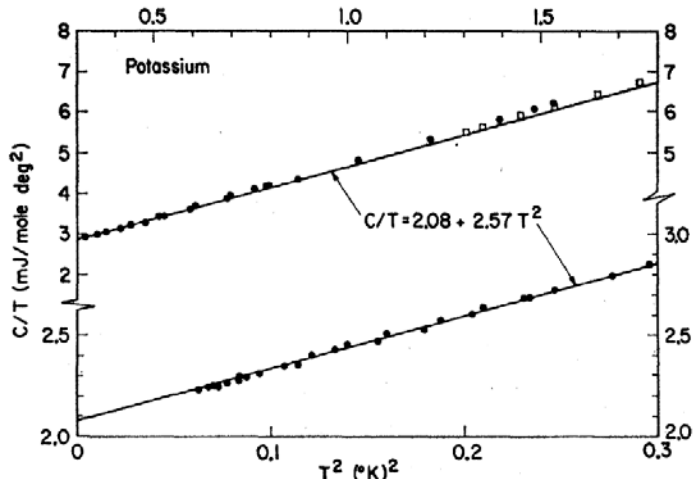
a. The external $\boldsymbol{E}$-field $\boldsymbol{E}=E\hat{z}$ is turned on, and the system comes to thermal equilibrium. Derive a formula for the energy $U(\boldsymbol{E},T)$.

b. The value of the field is 4.0 x 10^6 V/m. The temperature is 100K. What is the value of U/N in eV per molecule, or of U in J/mole? Make any appropriate approximations.

c. Find formulas for the polarization $\boldsymbol{P}$ (dipole moment per unit volume) and the dipolar susceptibility $\chi=(\partial P_z/\partial E_z)_{E\to 0}$

4. **Magnetization fluctuations**: Consider N moments μ in volume V, with two allowed orientations, $+\mu$ and $-\mu$, in an external field B at temperature T. Calculate the fluctuations of the magnetization $[<M^2>-<M>^2]^{1/2}/<M>$.

5. **Specific heat of a solid:** The figure (from W. H. Lien and N. E. Phillips, Phys. Rev. **133**, A1370 (1964)), shows the specific heat *C(T)* of potassium metal. The two curves refer to different horizontal temperature scales (see top and bottom axes.) The data are plotted as *C/T* vertically and T^2 horizontally. The temperature range measured is 0.25K<*T*<1.35K). There are two types of contributions to *C(T)* in a metal, from electrons and from phonons ("acoustic" vibrations).



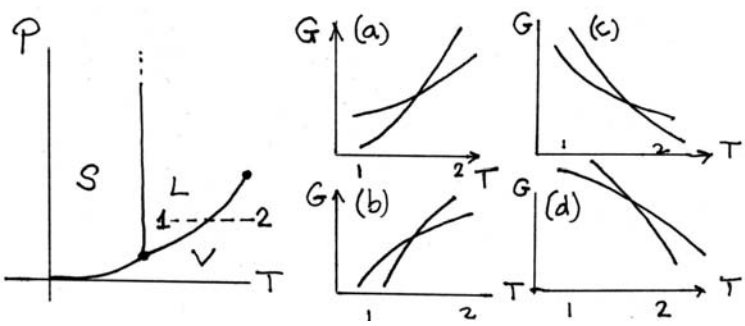
a. What is the power law *p* (as in $C \sim T^p$) for the electronic contribution C_{el} and the vibrational contribution C_{vib}?

b. What is the total electronic contribution to *C* extrapolated to 300K? What value C_{cl} would C_{el} have in a purely classical theory?

c. To what temperature T_x would you have to extrapolate so that $C_{el}=C_{cl}$? What is the interpretation of the temperature T_x?

d. Write a formula for T_x. You are not required in this problem to do any complete treatment of the theory of *C(T)*, just to give physically sensible answers. From your formula, compute a sensible theoretical value of T_x given that the electron concentration in potassium is 1.40×10^{22} cm^{-3}.

6. Liquid-Vapor coexistence: The dashed line on the *PT* phase diagram crosses the liquid-vapor coexistence curve (from 1 to 2) at constant *P*. Only one of the four schematic Gibbs energy diagrams a,b,c,d is a possible representation of liquid and vapor Gibbs energies along the path 1→2.



a. Which one and why?

b. What does the difference of slopes (on the *G vs. T* graph) at the crossing point tell you?