

- (1)  $\vec{R} = \sum_{i=1}^N \vec{r}_i$  and  $\langle \vec{r}_i \cdot \vec{r}_j \rangle = \langle \vec{r}_i \rangle \cdot \langle \vec{r}_j \rangle$  if  $i \neq j$   
 (since steps are random - i.e. not correlated.)  
 $\langle \vec{r}_i \rangle = 0$  by triangular symmetry.  
 $\langle \vec{r}_i \cdot \vec{r}_j \rangle = \delta_{ij}$   
 $\langle R^2 \rangle = \langle \sum_{ij} r_i \cdot r_j \rangle = \langle \xi^2 \rangle = N$   
 $\langle R^2 \rangle = N$  quite general for a random walk.

(2)  $D(\epsilon) = \sum_k \delta(\epsilon - E_k) \rightarrow N \frac{2\pi}{E_0} \left( \frac{\epsilon}{E_0} \right)^{1/2}$  in  $d=3$   
 where  $E_0 = \frac{\hbar^2 k_0^2}{2m}$  and  $k_0 = \frac{3\pi(N/V)^{1/3}}{V}$  in  $d=3$ .  
 inserting into the formula, we get  
 $D(\epsilon) \rightarrow \left( \frac{V}{2\pi} \right)^3 \cdot \frac{4\pi m}{\hbar^2} \left( \frac{2m\epsilon}{\hbar^2} \right)^{1/2}$

(a) derive this in  $d=3$

$$\begin{aligned} D(\epsilon) &= \left( \frac{V}{2\pi} \right)^3 \int d^3k \delta(\epsilon - E_k) = \left( \frac{V}{2\pi} \right)^3 4\pi \int_0^\infty k^2 dk \delta\left(\frac{\hbar^2 k^2}{2m} - \epsilon\right) \\ &= \left( \frac{V}{2\pi} \right)^3 4\pi \int_0^\infty k^2 dk \frac{\delta(k - \sqrt{2m\epsilon/\hbar^2})}{\hbar^2 k/m} \\ &= \left( \frac{V}{2\pi} \right)^3 \cdot \frac{4\pi m}{\hbar^2} \sqrt{\frac{2m\epsilon}{\hbar^2}} \quad \text{④ ED} \end{aligned}$$

(b)  $N(\mu, T) = 2 \sum_k \frac{1}{e^{\beta(E_k - \mu)} - 1}$  (2 for spin degeneracy)

$$\begin{aligned} N(\mu, T) &= 2 \int d\epsilon D(\epsilon) \frac{1}{e^{\beta(\epsilon - \mu)} - 1} \\ &= \frac{\pi^{\alpha/2} N}{\Gamma(\alpha/2)} \int \frac{dE}{E_0} \left( \frac{\epsilon}{E_0} \right)^{\alpha/2 - 1} \frac{1}{e^{\beta(\epsilon - \mu)} - 1} \end{aligned}$$

(c) At small  $x$ ,  $\int_0^\infty dx \frac{x^p}{e^{x-1}} \rightarrow \int_0^\infty dx x^{p-1} \rightarrow \infty$  if  $p \leq 0$

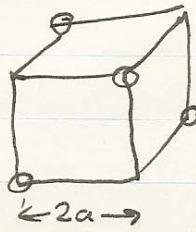
converges for  $p > 0$

(d) The maximum  $\mu$  is 0 and the maximum

$$\begin{aligned} N(\mu, T) &= \frac{\pi^{d/2}}{\Gamma(d/2)} 2N \int \frac{dE}{E_0} \left( \frac{\epsilon}{E_0} \right)^{d/2 - 1} \frac{1}{e^{\beta(\epsilon - \mu)} - 1} \\ &= N_{\max} = \left( \frac{kT}{E_0} \right)^{d/2} \times \text{const} \times \int_0^\infty dx \frac{x^{d/2 - 1}}{e^{x-1}} \end{aligned}$$

For  $d > 2$ ,  $N_{\max}$  is finite and decreasing to 0 at low  $T$ , which is Bose condensation. For  $d \leq 2$ ,  $N_{\max} = \infty$ . No condensate

3.



A:

The six possible states can be labeled by the face of the cube. If the field  $\vec{E}$  is in the  $z$  direction, it raises the energy of the  $-\hat{z}$  face state, by  $2eaE$ , and lowers the energy of the  $+\hat{z}$  face by the same amount. The other 4 states have energy 0

$$Z = 4 + 2 \cosh(\beta \cdot 2eaE) = 4e^{-\beta F}$$

$$U = F + TS = F - T \frac{\partial F}{\partial T} = \frac{\partial}{\partial \beta} (\beta F) = -\frac{\partial}{\partial \beta} \ln Z$$

$$(a) \quad U = \frac{-2eaE e^{\beta 2eaE} + 2eaE e^{-\beta 2eaE}}{4 + 2 \cosh \beta \cdot 2eaE}$$

$$U = -2eaE \left( \frac{2 \sinh 2eaE/kT}{4 + 2 \cosh 2eaE/kT} \right)$$

$$(b) \quad \text{evaluate } \epsilon_0 = 2eaE \text{ when } E = 10^6 \text{ V/m}$$

$$\epsilon_0 = 2(1.6 \times 10^{-19} \text{ C})(10^{-10} \text{ m})(10^6 \text{ V/m}) \\ = 3.2 \times 10^{-23} \text{ J}$$

This is small compared to  $k_B T$  with  $T=100 \text{ K}$

$$k_B T = 1.38 \times 10^{-21} \text{ J} \text{ so } \sinh x \rightarrow x, \cosh x \rightarrow 1$$

$$U \approx \frac{8(eaE)^2}{6kT} = \frac{8}{3} \frac{(3.2 \times 10^{-23} \text{ J})^2}{1.38 \times 10^{-21} \text{ J}} \cdot \frac{1}{1.602 \times 10^{-19} \text{ J/eV}} \\ = \underline{\underline{1.5 \times 10^{-6} \text{ eV}}}$$

$$U = \left( \frac{\mu^2}{3kT} \right) E^2 \quad (\mu = 2ea)$$

$$(c) \quad P = \frac{\partial U}{\partial E} = \frac{2}{3} \frac{\mu^2}{kT} E \quad \chi = \frac{2}{3} \frac{\mu^2}{kT}$$

4) Magnetization Fluctuations

$$Z = (e^{\beta \mu B} + e^{-\beta \mu B})^N = 2^N (\cosh \beta \mu B)^N$$

$$U = -N \langle \mu \rangle B = -\frac{\partial}{\partial \beta} \ln Z$$

$$\langle M \rangle = \frac{N}{V} \langle \mu \rangle = \frac{1}{BV} \frac{\partial}{\partial \beta} \ln Z = \frac{N}{BV} \frac{\partial}{\partial \beta} \ln \cosh(\beta \mu B)$$

$$\boxed{\langle M \rangle = \frac{N \mu}{V} \tanh \beta \mu B}$$

$$\langle M^2 \rangle - \langle M \rangle^2 = \left( \frac{1}{BV} \right)^2 (\langle E^2 \rangle - \langle E \rangle^2) = \frac{1}{B^2 V^2} \frac{\partial^2}{\partial \beta^2} \ln Z$$

$$= \frac{1}{BV} \frac{\partial}{\partial \beta} \langle M \rangle = \frac{1}{BV} \frac{N \mu}{V} \frac{\partial}{\partial \beta} \tanh \beta \mu B$$

$$= \frac{N \mu^2}{V^2} \frac{1}{\cosh^2 \beta \mu B}$$

$$\sqrt{\frac{\langle M^2 \rangle - \langle M \rangle^2}{\langle M \rangle}} = \frac{1}{\sqrt{N}} \frac{1}{\cosh \beta \mu B} \tanh \beta \mu B$$

$$= \frac{1}{\sqrt{N}} \frac{1}{\sinh \beta \mu B}$$

$$5. \quad a) \left( \frac{C_{el} \propto T^{-1}}{T \ll T_F} \right) \left( \frac{C_{vib} \propto T^3}{T \ll \Theta_D} \right)$$

$$b) \quad C_{el} = 2.08 \frac{\text{mJ}}{\text{mole K}^2} \cdot 300\text{K} = 0.624 \frac{\text{J}}{\text{mole K}}$$

classically  $C = \frac{3}{2} N k_B$  for  $N$  particles.  
for one mole  $N \rightarrow N_A$

$$C_{cl} = \frac{3}{2} N_A k_B = \frac{3}{2} (6.02 \times 10^{23}) \left( 1.38 \times 10^{-23} \frac{\text{J}}{\text{K}} \right) \\ = 12.5 \frac{\text{J}}{\text{mole K}}$$

$$c) \quad C_{el}(300\text{K}) \cdot \frac{T_x}{300\text{K}} = C_{cl}$$

$$T_x = (300\text{K}) \left( \frac{C_{cl}}{C_{el}} \right) = (300\text{K}) \left( \frac{0.624}{12.5} \right)^{-1} =$$

$$T_x = 6000\text{K}$$

This is the degeneracy temperature  
of the Fermi gas, approximately the  
Fermi temperature  $E_F/k_B$ .

$$d) \quad E_F = \frac{\hbar^2 k_F^2}{2m} \text{ and } (3\pi^2 n)^{1/3} = k_F \quad \text{since } \frac{2.4\pi}{3} k_F^3 \frac{V}{(2\pi)^3} = N$$

$$T_x \sim \frac{1}{k_B} \frac{\hbar^2}{2m} (3\pi^2 n)^{2/3}$$

$$= \frac{1}{1.38 \times 10^{-23} \frac{\text{J}}{\text{K}}} \frac{(1.05 \times 10^{-34} \text{ Js})^2}{2(0.911 \times 10^{-30} \text{ kg})} \underbrace{[(3\pi^2)(1.40 \times 10^{28} \text{ m}^{-3})]^{2/3}}_{5.56 \times 10^{19} \text{ m}^{-2}}$$

$$T_x = 25,000\text{ K}$$

right order of magnitude

## 6. Liquid - Vapor coexistence

at coexistence  $G_L = G_V$

away from coexistence the lower value of  $G$  wins.

$$dG = -SdT + Vdp$$

but  $p$  is constant.  $S = -\frac{\partial G}{\partial T}$  should be  $> 0$

(a) so, expect  $\frac{\partial G}{\partial T} < 0$  as in (c) and (d)

$$\text{now } C_p = T \left( \frac{\partial S}{\partial T} \right)_p = -T \frac{\partial^2 G}{\partial T^2} > 0$$

so the curvature of  $G$  must be negative  
as in (b) and (d)

therefore (d) is the only one that works.

(b) negative of slope =  $S$

difference of slope =  $-\Delta S$

latent heat =  $T \Delta S$

$$= -T \times \text{difference in slopes.}$$

The higher  $T$  phase has the more negative slope,  
therefore the higher  $S$ . Heat  $L = T \Delta S$  has to  
be added to make the liquid vaporize.