Physics 555	Fall 2007	HW 7

Due Friday December 7 **I. Screening and the Coulomb interaction**

1. Ziman derives the dielectric function explicitly only for the case of the free electron gas, where electron orbitals are plane waves. For the case of a real crystal, the orbitals are Bloch states $\psi_{\boldsymbol{k}n}=|\boldsymbol{k}n>$. Show that the corresponding SCF dielectric function is

$$\varepsilon(\vec{q},\omega)=1+\frac{4\pi e^2}{q^2}\sum_{\vec{k},n,n'}\frac{\left|\left\langle\vec{k}+\vec{q}n'\left|e^{i\vec{q}\cdot\vec{r}}\right|\vec{k}n'\right\rangle\right|^2\left[f\left(\vec{k}n\right)-f\left(\vec{k}+\vec{q}n\right)\right]}{\varepsilon\left(\vec{k}+\vec{q}n'\right)-\varepsilon\left(\vec{k}n\right)-\hbar(\omega+i\alpha)}$$

2. Show how to get the Fermi-Thomas limit ($\omega=0$ and $\boldsymbol{q}$ small, $T=0$) with no further approximation, for real, not free electrons, and exhibit the formula.

II. Creation and Destruction operators ("2nd Quantization")

3. For a system of massless Bosons (photons, phonons) the relevant space of states is spanned by the occupation number states, $\left|n_1,\ldots,n_i,\ldots\right\rangle$, where n_i is a (non-negative) integer interpreted as the number of Bose particles in the ith mode (each boson in the ith mode brings an energy $\hbar\omega_i$.) Creation and destruction operators are defined by

$$a_i\left|n_1,\ldots,n_i,\ldots\right\rangle=\sqrt{n_i}\left|n_1,\ldots,n_i-1,\ldots\right\rangle$$
$$a_i^+\left|n_1,\ldots,n_i,\ldots\right\rangle=\sqrt{n_i+1}\left|n_1,\ldots,n_i+1,\ldots\right\rangle$$

In thermal equilibrium, the probability $P(n_1,\ldots,n_i,\ldots)$ of the system being in the state $\left|n_1,\ldots,n_i,\ldots\right\rangle$ is

$$P(n_1,\ldots,n_i,\ldots)=\prod_i e^{-n_i\beta\hbar\omega_i}/Z_i$$

where the partition function Z_i of the ith mode is the usual grand canonical ensemble result, $Z_i=\prod_n\exp(-n\beta\hbar\omega_i)=[1-\exp(-\beta\hbar\omega_i)]^{-1}$. The operator $\hat{n}_i\equiv a_i^+a_i$ is the "number operator", because the state $\left|n_1,\ldots,n_i,\ldots\right\rangle$ is an eigenstates of $\hat{n}_i$ with eigenvalue n_i. Use the definitions above to prove that the thermal average occupancy is $\left\langle\hat{n}_i\right\rangle=[\exp(\beta\hbar\omega_i)-1]^{-1}$, the usual Bose-Einstein function. Also evaluate the fluctuation $\left\langle\hat{n}_i^2\right\rangle-\left\langle\hat{n}_i\right\rangle^2$. Find the value of $\left\langle\hat{n}_i\right\rangle$ and $\left\langle\hat{n}_i^2\right\rangle-\left\langle\hat{n}_i\right\rangle^2$ at the temperature where $\beta\hbar\omega_i=1$.

4. For massive Fermions, the same definitions work with a modification. The space of states is again spanned by the basis states $\left|n_1,\ldots,n_i,\ldots\right\rangle$, with the additional restriction that the integers n_i must be either 0 or 1. In a physical system of electrons, the total number is a fixed conserved quantity, but it is convenient to work in the grand canonical ensemble where the number is arbitrary, being determined by the chemical potential μ of a bath of particles in contact with the system. The space of all states $\left|n_1,\ldots,n_i,\ldots\right\rangle$, unrestricted as to total occupancy, is called "Fock space." The definition of creation and destruction operators is

$$c_i\left|n_1,\ldots,n_i,\ldots\right\rangle = \sqrt{n_i}\left|n_1,\ldots,n_i-1,\ldots\right\rangle$$
$$c_i^+\left|n_1,\ldots,n_i,\ldots\right\rangle = \sqrt{1-n_i}\left|n_1,\ldots,n_i+1,\ldots\right\rangle$$

In thermal equilibrium, the probability $P(n_1,\ldots,n_i,\ldots)$ of the system being in the state $\left|n_1,\ldots,n_i,\ldots\right\rangle$ is

$$P(n_1,\ldots,n_i,\ldots) = \prod_i e^{-n_i\beta(\varepsilon_i-\mu)}/Z_i$$

where the partition function Z_i of the ith mode is the usual grand canonical ensemble result, $Z_i = 1+\exp[-\beta(\varepsilon_i-\mu)]$. Again, the operator $\hat{n}_i \equiv c_i^+ c_i$ is the “number operator”, because the state $\left|n_1,\ldots,n_i,\ldots\right\rangle$ is an eigenstates of $\hat{n}_i$ with eigenvalue n_i. Use the definitions above to prove that the thermal average occupancy is $\left\langle\hat{n}_i\right\rangle = [\exp(\beta(\varepsilon_i-\mu))+1]^{-1}$, the usual Fermi-Dirac function. Also evaluate the fluctuation $\left\langle\hat{n}_i^2\right\rangle - \left\langle\hat{n}_i\right\rangle^2$. Find the value of $\left\langle\hat{n}_i\right\rangle$ and $\left\langle\hat{n}_i^2\right\rangle - \left\langle\hat{n}_i\right\rangle^2$ at a temperature where $\beta(\varepsilon_i-\mu)=1$. Also find the general formula for the fluctuation $\left\langle\hat{N}^2\right\rangle - \left\langle\hat{N}\right\rangle^2$ in the total number of Fermions (electrons, for example) in a system with $\left\langle\hat{N}\right\rangle = N$ total Fermions, in the grand canonical ensemble defined by β,μ. The total number operator is $\hat{N} = \sum_i \hat{n}_i$. Show that $\sqrt{\left\langle\hat{N}^2\right\rangle - \left\langle\hat{N}\right\rangle^2}$ is small compared to N.