

**1. Cooper pair radius.** The spatial part of the wave-function of a Cooper pair is

$$\psi(\vec{r}) = (1/\sqrt{V}) \sum_k g(k) e^{i\vec{k}\cdot\vec{r}}$$

where  $\mathbf{r}$  is the relative coordinate  $\mathbf{r}_2 - \mathbf{r}_1$  of the opposite spin electrons of the pair. This is the pair of Cooper's problem, superimposed on the Frozen Fermi sea, with Pauli-principle restriction forcing  $k$  to lie outside the occupied region.

a. Show the second equality below follows from the first.

$$\langle r^2 \rangle \equiv \int d\vec{r} r^2 |\psi(\vec{r})|^2 / \int d\vec{r} |\psi(\vec{r})|^2 = \sum_k |\nabla_{\vec{k}} g(\vec{k})|^2 / \sum_k |g(\vec{k})|^2$$

b. Using the solution for  $g(k)$  found in class, evaluate (to lowest order) the Cooper-pair radius, for the case where  $\Delta \ll \Omega \ll \epsilon_F$ . If the binding energy  $\Delta$  is  $\sim 2$  meV (e.g., metallic Pb), what would be the approximate radius?

**2. Quasiparticle entropy.** There is an elegant statistical argument indicating that for any system of weakly interacting Fermions (even when not in equilibrium!) the entropy is a simple function of the distribution  $f_i$ . Note that  $f_i$  need not be equal to the Fermi-Dirac distribution, but does if the system is in equilibrium. A proof is in the Landau-Lifshitz text *Statistical Physics Part 1*. You do NOT need to read this unless you are curious. The formula is:

$$S/k_B = - \sum_i [f_i \ln f_i + (1 - f_i) \ln(1 - f_i)]$$

where the index  $i$  labels the different single-particle states. In the first part of this exercise, you are supposed to prove that this formula is equivalent to the one you can get from the common text-book derivations.

a. The usual derivation is for equilibrium with a temperature bath at temperature  $T$  and a particle bath at chemical potential  $\mu$ . The partition function is calculated separately for each state  $i$  of energy  $\epsilon_i$ , which can be either empty or full. Derive the total partition function, and from that, the free energy, and from that, the entropy. Verify that it agrees with the formula above, if  $f_i$  is the Fermi-Dirac distribution.

b. Verify that if you use only the formula above, for general  $f_i$ , you can derive the Fermi-Dirac distribution by asking "what form of  $f_i$  gives the maximum entropy at a fixed total Fermion number  $N = \sum f_i$  and total energy  $U = \sum \epsilon_i f_i$ ." You may need Lagrange multipliers.

**3. BCS density of states.** An electron gas has a density of states  $N(\xi)$  which takes the value  $N(0)$  at the Fermi energy  $\xi=0$ . The notation is that  $\xi_k = \epsilon_k - \mu$ . The BCS excitations have energy  $E_k^2 = \xi_k^2 + \Delta^2$ . We neglect the possible  $k$ -dependence of the gap. The energy  $E_k$  is the **positive** square root. To be consistent, we should then use a convention for the normal excitations that  $\xi_k = |\epsilon_k - \mu|$ . This is equivalent to the "hole convention" for hole-type states (i.e. a particle removed from a state below the Fermi energy costs a positive energy.) Now we can define the "density of excitations" which is similar to the density of states, but with the sign convention of

the hole language. The density of excitations in a normal metal is then, for positive  $\xi_k$ ,  $2N(0)$ . We neglect the energy dependence for excitation energies  $\xi_k$  small compared with the Fermi energy. Compensating the factor of 2, there are zero excitations at negative energies  $\xi_k$ .

**a.** In terms of the normal state  $N(0)$ , compute the BCS density of excitations ( $E$  is positive).

$$N_{\text{BCS}}(E) = \sum_k \delta(E - E_k).$$

Even though you may think of this as derived from a free-electron result ( $\xi_k = \hbar^2 k^2 / 2m - \varepsilon_F$ ), your answer will be also correct for any metallic electron spectrum as long as  $N_{\text{normal}}(\xi)$  varies with  $\xi$  on a scale slow compared with  $\Delta$ .

**b.** The entropy can then be written as

$$S/k_B = - \int dE N_{\text{BCS}}(E) [f(E) \ln f(E) + (1 - f(E)) \ln(1 - f(E))]$$

Derive the formula for the BCS superconducting state specific heat, expressed as an energy integral. What approximate  $T$ -dependence does this imply at low  $T$  (where you can neglect the  $T$ -dependence of  $\Delta$ )?

**4. Nodes in the gap.** Many interesting superconductors have been discovered in the last 25 years. Many of them have a layered structure. They can be regarded, to decent approximation, as two-dimensional systems (dispersion of quasiparticles in the “c” direction is small and is neglected compared with dispersion in the a-b plane.) Some such systems apparently have “nodes” in their  $k$ -dependent gap. The gap,  $\Delta(k)$ , is defined for quasiparticle states  $k$  near the Fermi “surface”, and depends only on the value of  $k$  at the Fermi surface. There is no dependence on  $k$  perpendicular to the Fermi surface (in the usual picture.) There are points  $k_p$  on the two-dimensional Fermi surface where  $\Delta(k_p) = 0$ . Consider a simple model where the Fermi surface is a circle centered at  $k = 0$ , of radius  $k_F = \sqrt{(2m\varepsilon_F)}/\hbar$ . The  $k$ -dependent gap is approximated as  $\Delta(k) = \Delta \cos(n\phi)$ , where  $\phi$  is an angle measured from the center of the “Fermi circle.” The gap vanishes at angles  $\phi_p = p \cdot 2\pi/n + \pi/2n$ , and alternates in sign in between, except for the “s-wave” case  $n=0$ . The cases  $n = 1, 2, \dots$  are p-wave, d-wave, etc. Near the nodal angles  $\phi_p$ , the quasiparticle excitations (for  $n \neq 0$ ) have energies  $E_k^2 = \xi_k^2 + (n(\phi - \phi_p)\Delta)^2$ . At the nodal points, the energies are just normal state energies  $|\xi_k|$ . Show that the BCS density of excitations is no longer in the form you derived above, but instead is proportional to the normal state density of states, reduced by a factor proportional to  $(E/\Delta)$ , for  $E$  small compared with  $\Delta$ . What does this do to the specific heat at low  $T$  (where you can take  $\Delta$  to be independent of  $T$ )?