The electron-phonon coupling constant λ *

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Tables of values of the electron-phonon coupling constants λ and $\lambda_{\rm tr}$ are given for selected elements and compounds. A brief summary of the theory is also given.

I. THEORETICAL INTRODUCTION

Grimvall [1] has written a review of electron-phonon effects in metals. Prominent among these effects is superconductivity. The BCS theory gives a relation $T_c \approx \Theta_D \exp(-1/N(0)V)$ for the superconducting transition temperature T_c in terms of the Debye temperature Θ_D. Values of T_c are tabulated in various places, the most complete prior to the high T_c era being ref. [2]. The electron-electron interaction V consists of the attractive electron-phonon-induced interaction minus the repulsive Coulomb interaction. The notation is used

$$\lambda = N(0)V_{\rm e-ph} \tag{1}$$

and the Coulomb repulsion $N(0)V_{\rm c}$ is called μ, so that $N(0)V = \lambda - \mu^*$, where μ^* is a "renormalized" Coulomb repulsion, reduced in value from μ to $\mu/[1+\mu\ln(\omega_P/\omega_D)]$. This suppression of the Coulomb repulsion is a result of the fact that the electron-phonon attraction is retarded in time by an amount $\Delta t \approx 1/\omega_D$ whereas the repulsive screened Coulomb interaction is retarded by a much smaller time, $\Delta t \approx 1/\omega_P$ where ω_P is the electronic plasma frequency. Therefore, μ^* is bounded above by $1/\ln(\omega_P/\omega_D)$ which for conventional metals should be ≤ 0.2.

Values of λ are known to range from ≤ 0.10 to ≥ 2.0. The same coupling constant λ appears in several other physical quantities, such as the electronic specific heat $C_{\rm el}(T)$ which at low temperature equals

$$C_{\rm el}(T) = \frac{\pi^2}{3} N(0)(1+\lambda)k_B^2 T. \tag{2}$$

This relation enables a value for $(1+\lambda)$ to be extracted if the bare density of states $N(0)$ is known from quasiparticle band theory. This proceedure has large uncertainties, so the resulting values of λ are subject to substantial error and are not tabulated here.

When the BCS ideas are carefully worked out using the actual electron-phonon interactions (Migdal-Eliashberg theory [3]) then a quite complicated but in principle solvable relation occurs between electron-phonon coupling and T_c. If the anisotropy of the superconducting gap is ignored (or washed out by non-magnetic impurity scattering) then theory simplifies and T_c depends on μ^* and a single function $\alpha^2F(\Omega)$ which is similar to the phonon-density of states $F(\Omega)$. α^2 contains an average square electron-phonon matrix element. Quasiparticle tunneling experiments in planar tunnel junction geometry [4,5] in principle provide a way of measuring this function, which is related to λ by

$$\lambda = 2\int_0^\infty \frac{d\Omega}{\Omega}\alpha^2F(\Omega). \tag{3}$$

This provides perhaps the most reliable known values for λ. The techniques and the data are reviewed in ref. [5]. However, there are still significant uncertainties in values of λ obtained this way, caused by difficulties in making good yet partially transparent barriers and ignorance of such details as the transmission coefficients for tunneling, inelastic effects in the barrier region, etc. For a few metals (principally Pb, In, Tl, and alloys of these) the barriers seem particularly clean or else the complexities somehow cancel out; the accuracy of the resulting $\alpha^2F(\Omega)$ is well tested through various self-consistency checks. Tunneling in point contact geometry gives valuable information about $\alpha^2F(\Omega)$, especially in materials with weaker electron-phonon interactions where the planar junction techniques do not work. However, the absolute values of λ obtained this way are rather variable and are not listed here.

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The Migdal-Eliashberg theory was solved numerically for T_c as a functional of $\alpha^2F(\Omega)$ and μ^* by McMillan [6]. He fitted his results to an approximate formula, generalizing the BCS result $T_c = \Theta_D \exp[-1/(\lambda - \mu^*)]$,

$$T_c = \frac{\Theta_D}{1.45} \exp\left[-\frac{1.04(1+\lambda)}{\lambda - \mu^*(1+0.62\lambda)}\right]. \tag{4}$$

The parameter μ^* is assigned a value in the range 0.10-0.15, consistent with tunneling and with theoretical guesses. The choice of μ^* is fairly arbitrary, but fortunately its precise value is not too important unless T_c is very low. Most of our knowledge of values of λ comes from using Eqn. (4) to extract a value of λ from measured values of T_c and Θ_D. Values deduced in this fashion are denoted $\lambda_{\rm McM}$ in this article. Subsequent to McMillan's work, experiments and further numerical studies [7] worked out the limits of applicability of Eqn. 4. The "prefactor" $\Theta_D/1.45$ works well only for elements or materials whose phonon density-of-states is similar in shape to elements like Nb. The correct [7] prefactor $\omega_{\ln}/1.20$, is not measureable except by experiments like tunneling. The definition of $\omega_{\ln}$ is

$$\omega_{\ln} = \exp\left[\frac{2}{\lambda}\int_0^\infty d\Omega \frac{\ln(\Omega)}{\Omega}\alpha^2F(\Omega)\right]. \tag{5}$$

Lack of knowledge of $\omega_{\ln}$ limits the accuracy of values of $\lambda_{\rm McM}$, especially in compound materials with complicated phonon dispersion. When T_c becomes reasonably large (T_c greater than $\approx 5\%$ of Θ_D or $\lambda \geq 1.2$) Eqn. (4) underestimates T_c. Approximate correction factors were given in ref. [7]. Unfortunately, additional parameters are required to give an accurate formula for T_c.

Calculations of μ or μ^* are computationally demanding and are not yet under theoretical control. Calculations of λ are slightly less demanding, are under somewhat better theoretical control, and have been attempted for many years. Prior to 1990, calculations of λ generally required knowing the phonon frequencies and eigenvectors as input information, and approximating the form of the electron-ion potential. Results of these calculations are not tabulated here. McMillan [6] and Hopfield [8] pointed out that one could define a simpler quantity,

$$\eta \equiv N(0)\langle I^2\rangle = M\langle\omega^2\rangle\lambda, \tag{6}$$

$$\langle\omega^2\rangle \equiv \frac{2}{\lambda}\int_0^\infty d\Omega\Omega\alpha^2F(\Omega). \tag{7}$$

The advantage of this is that η and $\langle I^2\rangle$ are purely "electronic" quantities, requiring no input information about phonon frequencies or eigenvectors. Gaspari and Gyorffy [9] then invented a simplified algorithm for calculating η, and many authors have used this. These calculations generally require a "rigid ion approximation" or some similar guess for the perturbing potential felt by electrons when an atom has moved. Given η, one can guess a value for $\langle\omega^2\rangle$ (for example, from Θ_D) and thereby produce an estimate for λ. Values produced this way are not tabulated in the present chapter. Instead the reader is referred to the following literature: Sigalas and Papaconstantopoulos [10] have given a recent tabulation for d-band elements, and Skriver and coworkers have published calculations for rare earths [11] and lanthanides [12] which are particularly valuable since usually no other estimate of λ is available. Brorson *et al.* [13] have extracted measured values of η from measured rates of thermal equilibration of hot electrons in various metals, using a theoretical relation [14]. Fairly recently, theory has progressed to the point where "first- principles" calculations [15] can be made of phonon dispersion curves using density functional theory, usually in "local density approximation" (LDA) for quite complicated systems. It is not too hard to extend these calculations to give $\alpha^2F(\Omega)$ and λ; these values should be "reproducible" in the sense that most theorists would agree upon a unique recipe. Such calculations have been done by Savrasov and Savrasov [16] and by Liu and Quong [17]. The results accord well with other methods of finding λ. Values of λ obtained this way will be denoted $\lambda_{\rm LDA}$.

The T-dependence of the electrical resistivity sometimes offers an accurate way of evaluating the electron-phonon coupling. In clean metals (defined by a large resistance ratio $\rho(300K)/\rho(T = T_c + \epsilon)$) the resistivity is normally dominated by electron-phonon interactions. Using the standard form $\sigma = 1/\rho = ne^2\tau/m$, the scattering rate $\hbar/\tau(T)$ at high temperatures is $2\pi\lambda_{\rm tr}k_BT$ which defines a coupling constant $\lambda_{\rm tr}$ which is very closely related to λ. Coulomb scattering of electrons with each other also contributes, but is smaller by the factor $(\mu^2/\lambda)N(0)k_BT$ which is usually $\approx 10^{-2}$ at room temperature. The derivation of this result depends on Bloch-Boltzmann transport theory, which is closely analogous to Migdal-Eliashberg theory of superconductivity. For both theories, the "Migdal theorem" shows that corrections (Feynman diagrams with phonon vertex corrections) should be smaller by a factor $N(0)\hbar\omega_D$. Both theories contain anisotropy corrections, which are almost always small. When anisotropy is ignored, superconductivity depends on the isotropic parameter λ and resistivity on the isotropic parameter $\lambda_{\rm tr}$. These two coupling constants are related to the electron-phonon matrix elements $M_{k,k'}$ and the phonon frequencies $\omega_{k-k'}$ by the formula

$$\lambda_{\rm w} = N(0)\frac{\sum_{k,k'} w(k,k')|M_{k,k'}|^2/\hbar\omega_{k-k'}\delta(\epsilon_k)\delta(\epsilon_{k'})}{\sum_{k,k'} w(k,k')\delta(\epsilon_k)\delta(\epsilon_{k'})} \tag{8}$$

where ϵ_k is the quasiparticle energy of an electron constrained by the δ-function to be at the Fermi energy, the labels k,k' are short for the electron quantum numbers (wavevector, band index, spin), and $w(k,k')$ is a weight function to be specified. The superconducting λ uses $w = 1$ for the weight function, while the transport $\lambda_{\rm tr}$ uses $w(k,k') = (v_{kx} - v_{k'x})^2$, where $\hbar v_{k,x}$ is the group velocity $\partial\epsilon_k/\partial k_x$. A review of the theory and the formulas is given in ref. [18]. Simultaneous calculations [16,19] of these two parameters are available for a number of metals, shown in Table I. From this it is known that the difference is typically no more than 15%, and never as much as a factor of 2. In principle there are separate coupling constants $\lambda_{\rm tr,xx}$ and $\lambda_{\rm tr,zz}$ in hexagonal or tetragonal metals, but the anisotropy of $\lambda_{\rm tr}$ is believed to be quite small. The explanation for the observed similarity between λ and $\lambda_{\rm tr}$ must be that the weight factor $(v_{kx} - v_{k'x})^2$ correlates only weakly with the matrix element $|M_{k,k'}|^2$. The same lack of correlation will also guarantee that the anisotropy of the resistivity tensor in non-cubic materials will derive primarily from the anistropy of the inverse effective mass tensor n/m rather than from anisotropy in $\lambda_{\rm tr}$.

To extract a value of $\lambda_{\rm tr}$ from $\rho(T)$ data, the safest proceedure is a three parameter fit to the data using the Bloch-Grüneisen formula,

$$\rho_{\rm BG} = \rho_0 + \frac{2\pi\lambda_{\rm tr}k_BT/\hbar}{(n/m)e^2}\int_0^{\omega_D}\frac{d\Omega}{\Omega}\left(\frac{\Omega}{\omega_D}\right)^4\left[\frac{\hbar\Omega/k_BT}{\sinh(\hbar\Omega/2k_BT)}\right]^2. \tag{9}$$

At high T the factor $[\,]^2$ becomes 4 and thus the integral becomes 1. The three fitting parameters are ρ_0, ω_D, and the ratio $\lambda_{\rm tr}/(n/m)$. This equation assumes either an isotropic polycrystalline sample or else cubic symmetry, and also assumes that the phonon dispersion is adequately represented by a Debye model. It is easy to generalize to anisotropic or non-Debye cases, at the cost of further fitting parameters. For example, the case of completely general phonon dispersion is handled simply by replacing the factor $2\lambda_{\rm tr}(\Omega/\omega_D)^4$ by a function $\alpha^2_{\rm tr}F(\Omega)$. Fortunately, except at low T, the form of $\rho_{\rm BG}$ is not very sensitive to the form of $\alpha^2_{\rm tr}F(\Omega)$. One exception is ReO_3 where the acoustic vibrations are mainly Re-like and low in energy, while the optic vibrations are mainly O-like, and very high in energy. For that material, it is adequate [20] to represent $\alpha^2_{\rm tr}F(\Omega)$ by a sum of a Debye and an Einstein piece, $2\lambda_D(\Omega/\omega_D)^4 + (\lambda_E\omega_E/2)\delta(\Omega - \omega_E)$, where $\lambda_{\rm tr}$ is $\lambda_D + \lambda_E$. The remaining problem is that the tensor parameter n/m must also be known in order to get a value for $\lambda_{\rm tr}$, and there is no firm experimental method. The theoretical formula is

$$\left(\frac{n}{m}\right)_{\alpha\beta} = \sum_k v_{k,\alpha}v_{k,\beta}\delta(\epsilon_k). \tag{10}$$

For cubic symmetry, the tensor is a scalar, $(n/m)_{\alpha\beta} = (n/m)\delta_{\alpha\beta}$. It is usually not useful or even possible to make separate definitions of a scalar n (because it is unclear how many of the valence electrons should be counted) or of a tensor or scalar m or $1/m$. Another way to write Eqn. (10) is $(n/m)_{\alpha\beta} = N(0)\langle v_\alpha v_\beta\rangle$ where the angular brackets denote a Fermi surface average. One can also define a "Drude plasma frequency" $\Omega^2_{\alpha\beta} = 4\pi e^2(n/m)_{\alpha\beta}$ which governs both dc and ac conductivity. Unfortunately it is not possible to get reliable values of the Drude plasma frequency from optical experiments, for a variety of reasons, outlined by Hopfield [21]. Surprisingly, it seems that LDA band theory gives for many metals very reliable values of $(n/m)_{\alpha\beta}$ which yield therefore good values of $\lambda_{\rm tr}$. In quite a few metals, this provides the best available estimate of λ.

There are several other ways of getting values of parameters related to λ. Quasiparticles near the Fermi surface have energies and lifetimes given by poles of the Green's function or zeros of $G^{-1}(k,\omega)$ for small $|\omega|$ in the complex ω-plane,

$$\begin{aligned} G^{-1}(k,\omega) &= \omega - \epsilon_k - \Sigma(k,\omega) \\ &= (1+\lambda_k(T))(\omega + i/2\tau_k(\omega,T)) - \epsilon_k \end{aligned} \tag{11}$$

where $\lambda_k(T)$ is defined as $-\partial\Sigma_1(k,\omega)/\partial\omega|_{\omega=0}$, $\lambda_k/2\tau_k$ is defined as $-\Sigma_2(k,\omega)$, and the zero of energy is shifted so that $\Sigma_1(k,\omega=0)$ is absorbed and disappears. Then $\lambda_k(T)$ is the mass enhancement parameter for the quasiparticle state k, and λ is the average of $\lambda_k(T=0)$ over the Fermi surface. The spectral weight function, -Im$G^{-1}(k,\omega)$ then has a Lorentzian resonance centered at the renormalized quasiparticle energy $E_k = \epsilon_k/(1+\lambda_k(T))$ with width equal to the lifetime broadening $1/\tau_k(E_k,T)$. If electron-phonon interactions dominate, then at $T \geq \Theta_D$, where $\lambda_k(T) \approx 0$, $\hbar/\tau_k(E_k,T) = 2\pi\lambda_k(T=0)k_BT$. This also assumes that the state k being probed has a small enough $|\epsilon_k|$ so that $|E_k| \leq k_B\Theta_D$.

Various resonance methods can measure either $E_k(T)$ and $1/\tau_k(T)$. As an example, cyclotron resonance can do this for extremal orbits on the Fermi surface, but generally only at low temperatures where $E_k = \epsilon_k/(1+\lambda_k(T=0))$. Under these conditions, $1/\tau_k(T)$ is small and does not contain complete information about λ_k; an orbit-averaged value of $(1+\lambda)$ can be extracted using theoretical values of the unrenormalized quasiparticle band structure ϵ_k. In principle, photoemission spectroscopy could measure $1/\tau_k(T)$ at high T, and thus directly measure λ_k with no need for theoretical input. The principal difficulty is that the perpendicular component $k_\perp$ of the wavevector $\vec{k}$ can not be directly measured; a measured spectrum is a superposition of spectra for a range of values of $k_\perp$. One case where this is avoided is for a surface state. If there is no bulk state for some range of energy and some $\vec{k}_\parallel$, then a surface state may occur in that energy range with a sharp two-dimensional k-vector. Then high-resolution photoemission can measure $1/\tau_k$ and thus λ_k for that state.

II. COMMENTARY ON λ VALUES

Tables II through V contain values of λ for selected materials. I have tabulated only those cases where it seems likely that the value will not change greatly with time. Therefore, many interesting material are omitted, such as the cuprate, bismuthate, and fullerene superconductors. Especially in the cuprates, properties of quasiparticles, including even whether they exist and what spin and charge they carry, are still mysterious. Electron-phonon coupling seems to affect some properties of cuprates, yet seems to be missing from other properties. My opinion is that Migdal-Eliashberg theory and Eqn. (4) do not apply to cuprates, but probably do apply to fullerenes and to the $BaBiO_3$ family of superconductors. Values of λ of order 1 probably apply to the latter two families, but firm numbers are hard to spot.

Table II contains elements which are superconducting in crystalline form at atmospheric pressure, including two cases (Ga and La) where there is information about a metastable phase. The entries are taken from previously cited sources plus refs. [22]- [33]. The superconducting elements provide the best opportunity to compare λ values obtained by different methods. In particular, for Pb, V, Nb, and Ta there are transport, tunneling, and LDA values in addition to values from the McMillan equation, Eq. (4). In all cases, McMillan values are less than all other values, strongly suggesting that the McMillan equation underestimates λ. There are two main causes of this: (1) the McMillan prefactor $\Theta_D/1.45$ is often larger than the correct prefactor $\omega_{\ln}/1.2$, although although probably not by much for the elements V, Nb, and Ta because the McMillan equation was based on the phonon spectrum of Nb; (2) knowledge of μ^* is still primitive; it is assumed to be $0.10-0.13$ but Table II suggests that it is often larger. In particular, in V it is suspected that spin-fluctuation effects may increase μ^* above the "renormalized" value $\mu/[1+\mu\ln(\omega_P/\omega_D)]$ because the characteristic spin-fluctuation frequencies are lower than ω_P. For very strong-coupling materials like Pb, the McMillan equation should overestimate λ because in this regime the McMillan equation is known to underestimate T_c. However, in Pb the other causes must dominate since $\lambda_{\rm McM}$ is an underestimate. I believe that theoretical LDA values are now more reliable than McMillan values. Tunneling values are excellent for the "simple" metals β-Ga, In, Sn, Hg, Tl, and Pb, but require more complicated junction preparation methods and more complicated theoretical analyses for d-band and exotic materials, which degrades the believability of the numbers. For most elements I think the transport values are good, but Tl and Re may have too high values of $\lambda_{\rm tr}$.

Table III shows values for four elements where tunneling experiments [34,35] have been done on superconducting amorphous phases. Bi is semimetallic and not superconducting in its crystalline phase. The divergent experimental values of λ may reflect differences between samples of amorphous Bi made by different proceedures, or may instead reflect difficulties in accurate measurement of tunneling characteristics, especially at bias voltages near the superconducting gap where λ can have important contributions due to soft vibrations which are hard to measure accurately. In Ga and Sn (but not in Pb) the amorphous phase has enhanced values of T_c. In all three elements, λ is enhanced. The reason is the softening of the vibrational spectrum in amorphous phases, which raises λ more than T_c.

Table IV gives values for crystalline elements which are non-superconducting, using information from previously cited sources plus refs. ([36]- [38]). Values from transport are known for many elements and appear consistent with other values when available. It would be very interesting to have reliable values for ferromagnetic elements. However, one expects that there should be quite different values of λ for the up and the down spin species, but two values ($\lambda_\uparrow$ and $\lambda_\downarrow$) cannot be extracted independently from transport measurements. Therefore we must wait for theoretical values and measurements by spin-sensitive techniques. As an example, Gd is ferromagnetic with a Curie temperature 292K. Photoemission below this temperature has seen a surface state carrying the majority ($\uparrow$) spin, and its λ value is listed.

Table V lists most of the intermetallic compounds where there is reliable information [39]- [50]. Evaluating $\lambda_{\rm tr}$ for compounds is often risky. There are two main difficulties: (1) the measured $\rho(T)$ is sometimes sample dependent, especially in oxide materials where polycrystalline bulk, single crystal, and thin film samples give different results; it

is hard to know the "true" $\rho(T)$, but usually safe to assume that the smallest values are the best; (2) it is important to be sure that the transport mean free path is at least $10\mathring{A}$; otherwise, wavevector $\vec{k}$ is not a good quantum number, Bloch-Boltzmann theory is not applicable, and $\lambda_{\rm tr}$ is ill-defined. In Eqn. (8), the group velocity $\hbar v_{k,x} = \partial\epsilon/\partial k_x$ is only well defined when wavevector $\vec{k}$ is a good quantum number. The superconducting λ, however, is still well defined from Eqn. (8) provided the quantum numbers k, k' are reinterpreted as labels for the exact eigenstates of the disordered material.

RuO_2 is an interesting case where the transport value of λ suggests the possible occurrence of superconductivity at not too low a temperature, and experiment may not have tested below helium temperature. The layered intermetallic borocarbides have gotten a lot of attention recently. In the case of $LuNi_2B_2C$, McMillan and transport values agree which suggests (as does other evidence) conventional superconductivity. For the related material $La_3Ni_2B_2N_{3-\delta}$, transport and McMillan values badly disagree. My guess is that the transport value may change with better data, but usually the transport value goes down in time as samples improve, rather than up in time. An alternate possibility is the occurrence of unconventional superconductivity. A similar situation holds for cuprates, where transport values of λ of order 1 are found using LDA bands, whereas to account for T_c near 100K, one needs λ of order 3 or higher. There are clear signs of unconventional behavior, which in my mind invalidates the transport analysis based on Bloch-Boltzmann theory. Another interesting intermetallic, Sr_2RuO_4, is also not listed. Nice-looking resistivity measurements are now being published, but do not accord well with the Bloch-Grüneisen formula, which is another sign of unconventional behavior. It is not clear whether the concept of an electron-phonon λ value can be retained in such cases. In most materials where the confusion is large, I have not tabulated any λ values, but $La_3Ni_2B_2N_{3-\delta}$ is included as a sign of the potential hazards.

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TABLE I. Comparison of theoretical values of λ and $\lambda_{\rm tr}$

Metal	$\lambda_{\rm theor}$	$\lambda_{\rm tr,theor}$
Al[a]	0.44	0.37
Pb[a]	1.68	1.19
V[a]	1.19	1.15
Nb[a]	1.26	1.17
Nb[b]	1.12	1.07
Ta[a]	0.86	0.83
Ta[b]	0.88	0.57
Mo[a]	0.42	0.35
Cu[a]	0.14	0.13
Cu[b]	0.111	0.116
Pd[a]	0.35	0.43
Pd[b]	0.41	0.46

[a]Full LDA theory, Ref. [16].
[b]LDA energy bands, rigid ion approximation, experimental phonons, Ref. [19].

TABLE II. Values of λ for superconducting crystalline elements

Metal	T_c(K)	Θ_D(K)	λ_{McM}[a]	λ_{tr}	λ_{tun}	λ_{LDA}
Be	0.026	1390	0.23			
Al	1.16	428	0.38	0.39[b]		0.44[c] [d]
Zn	0.85	309	0.38	0.46[e]		
α-Ga	1.08	325	0.40			
β-Ga	5.9				0.97[f]	
Cd	0.52	209	0.38	0.37[e]		
In	3.40	112	0.69		0.834[g]	
Sn	3.72	200	0.60		0.72[h]	
Hg	4.16	72	1.00		1.60[i]	
Tl	2.38	79	0.71	1.11[e]	0.78[f]	
Pb	7.19	105	1.12	1.48[b]	1.55[j]	1.20[d],1.68[c]
Ti	0.39	425	0.38	0.50[k]		
V	5.30	399	0.60	1.09[b]	0.83[l]	1.19[c]
Zr	0.55	290	0.41	0.55[k]		
Nb	9.22	277	0.82	1.06[b]	1.05[m]	1.26[c]
Mo	0.92	460	0.41	0.32[b]		0.42[c]
Ru	0.49	550	0.38	0.45[e]		
Hf	0.09	252	0.34	0.42[k]		
Ta	4.48	258	0.65	0.87[b]	0.69[n], 0.73[o]	0.86[c]
W	0.012	390	0.28	0.26[b]		
Re	1.69	415	0.46	0.76[e]		
Os	0.65	500	0.39	0.54[e]		
Ir	0.14	420	0.34	0.50[b]		
α-La	4.88	151	0.81[p]		(0.77)[q]	
β-La	6.00	139	0.93[p]			
Th	1.38	165	0.56[b]	0.52[b]		

[a]Unless otherwise noted, from Ref. [6].
[b]Ref. [22].
[c]Ref. [16].
[d]Ref. [17].
[e]Ref. [23].
[f]Ref. [24].
[g]Ref. [25].
[h]Ref. [26].
[i]Ref. [27].
[j]Ref. [28].
[k]Ref. [18].
[l]Ref. [29].
[m]Ref. [30].
[n]Ref. [31].
[o]Ref. [32].
[p]from Eqn. 4 using data from Ref. [2].
[q]Difficulties with the junction required *ad hoc* modifications in the analysis, Ref. [33].

TABLE III. Values of λ for superconducting amorphous elements

Metal	T_c	λ_{tun}
Ga	8.56	2.25[a]
Sn	4.5	0.84[b]
Pb	7.2	1.91[a]
Bi	6.1	2.46[a],1.84[b]

[a]Ref. [34].
[b]Ref. [35].

TABLE IV. Values of λ for non-superconducting crystalline elements

Metal	$\lambda_{\rm tr}$ [a]	$\lambda_{\rm LDA}$	$\lambda_{\rm other}$
Li	0.35	0.45 – 0.51 [b]	
Na	0.14		0.24 [c]
K	0.11		
Rb	0.15		
Cs	0.16		
Cu	0.13	0.14 [d]	0.14±0.02 [e]
Ag	0.12		
Au	0.15		0.2 [f]
Mg	0.20 [g]		
Ca	0.05		
Ba	0.27		
Sc	0.51 [g]		
Y	0.62 [g]		
Pd	0.47	0.35 [d]	
Pt	0.66		
Gd			0.6 [h]

[a] Unless otherwise noted, from Ref. [22].
[b] Ref. [17].
[c] Surface quantum well state, Ref. [36].
[d] Ref. [16].
[e] Surface state on Cu(111), Ref. [37].
[f] Extrapolation from superconducting alloys, Ref. [38].
[g] Ref. [23].
[h] Surface state with quantum numbers $5d(z^2,\uparrow)$ on Gd(0001), P. D. Johnson, private communication.

TABLE V. Values of λ for crystalline compounds and ordered intermetallics

Metal	T_c	λ_{tun}	λ_{tr}	λ_{McM}
In_2Bi	5.6	1.40[a]		
Bi_2Tl	6.4	1.63[a]		
Tl_7Sb_2	5.2	1.43[a]		
V_3Si	17.1	0.89[b]		
Nb_3Al	18.5	1.7[c]		
Nb_3Sn	17.8	1.75[d]		
Nb_3Ge	$\approx$20	1.7[b]		
NbN	16.0	1.46[b]		
NbO	1.4		0.51[e]	0.41[e]
ReO_3	$<$0.02		0.35[f]	
RuO_2	$<$4.2		0.5 ± 0.1 [g]	
$CoSi_2$	1.22		0.44[f]	
Pd_2Si			0.15 – 0.20[f]	
$LuNi_2B_2C$	16.1		0.9[h], 0.8[i]	1.0[h]
$La_3Ni_2B_2N_{3-\delta}$	12.25		0.29[j]	0.86[j]

[a] Ref. [39].
[b] Ref. [40].
[c] Ref. [41].
[d] Ref. [30,42].
[e] Ref. [43].
[f] Ref. [20].
[g] Ref. [44].
[h] Using data from Ref. [45] and theory from Refs. [46] and [47].
[i] Ref. [48].
[j] Using data from Ref. [49] and theory from Ref. [50].