Physics 555 Fall 2003

Solid State Physics

Problem Set # 8, Due Friday Dec. 12

Reminder: Final exam Monday Dec. 15, 8:00-11:30am

1. **Cooper pair radius.**
   The radial part of the wave-function of a Cooper pair is

$$\psi(\vec{r}) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} g(\vec{k}) e^{i\vec{k}\cdot\vec{r}}$$

   where $\vec{r}$ is the relative coordinate $\vec{r}_2 - \vec{r}_1$ of the opposite spin electrons of the pair.

   a. Show that

$$< r^2 > \equiv \int d^3r \; r^2 |\psi(\vec{r})|^2 = \sum_{\vec{k}} |\vec{\nabla}_{\vec{k}} g(\vec{k})|^2.$$

   b. Use the solution found in class,

$$g(\vec{k}) = \begin{cases} \frac{C}{2\epsilon_k + \Delta} & \text{when } 0 < \epsilon_k < \Theta \\ 0 & \text{otherwise} \end{cases}$$

   where Δ is the binding energy, $\epsilon_k = \hbar^2 k^2/2m - \epsilon_F$, and C is a normalization constant, to get the result (valid for $\Delta \ll \Theta$)

$$\xi \equiv < r^2 >^{\frac{1}{2}} \approx c \frac{\hbar v_F}{\Delta}.$$

   Evaluate the coefficient c. Note that $k_F \xi \approx \epsilon_F/\Delta$, which is a large number, typically 10^4.

2. **Fermi-Dirac Distribution** As practice using creation and destruction operators, derive the Fermi-Dirac distribution in the following way. First we define a particular version of an average. Let $< \hat{A} >_0$ mean the canonical thermal average expectation of the operator $\hat{A}$ in an equilibrium system defined by a Hamiltonian $\mathcal{H}_0$, that is

$$< \hat{A} >_0 = \frac{\mathrm{tr} \hat{A} e^{-\beta(\mathcal{H}_0 - \mu\hat{N})}}{\mathrm{tr} e^{-\beta(\mathcal{H}_0 - \mu\hat{N})}}.$$

   Let $\mathcal{H}_0$ be a single particle system $\mathcal{H}_0 = \sum_i \epsilon_i \hat{c}_i^\dagger c_i$ whose single-particle states are labeled by quantum numbers $i = (\ldots)$ where the list includes the $\hat{z}$ component of spin in some convenient coordinate system. The trace operation goes over the states $|n_1, n_2, \ldots >$ of $\mathcal{H}_0$. Show that $< c_1^\dagger c_2 >= 0$ (where states 1 and 2 are different, and that $< c_1^\dagger c_1 >= f_1$ (where f_1 is the Fermi-Dirac distribution.)

3. **Electron-electron interaction**

For the same system, evaluate $< \hat{V} >_0$, where

$$\hat{V} = \frac{1}{2} \sum_{abcd} < ab|v|cd > c^{\dagger}_{a\sigma} c_{b\sigma} c^{\dagger}_{c\sigma'} c_{d\sigma'}.$$

That is, write an expression for $< \hat{V} >_0$ in terms of matrix elements $< ab|v|cd >$ and other defined quantities such as single-particle energies ϵ_a, single-particle equilibrium distributions f_a etc. Write out the $\vec{r}$-space integral form of the required matrix elements. Explain the relation to Hartree-Fock theory (in general, this is not the full Hartree-Fock energy.) Notice the role of spin in the exchange term.