Physics 555 Fall 2003

Solid State Physics

Problem Set # 8, Due Friday Dec. 12

Reminder: Final exam Monday Dec. 15, 8:00-11:30am

1. Cooper pair radius.
   The radial part of the wave-function of a Cooper pair is
   \[ \psi(\vec{r}) = \frac{1}{\sqrt{V}} \sum_k g(k) e^{i\vec{k} \cdot \vec{r}} \]
   where \( \vec{r} \) is the relative coordinate \( \vec{r}_2 - \vec{r}_1 \) of the opposite spin electrons of the pair.

   a. Show that
   \[ < r^2 > \equiv \int d^3r \ r^2 |\psi(\vec{r})|^2 = \sum_k |\vec{\nabla}_k g(k)|^2. \]

   b. Use the solution found in class,
   \[ g(k) = \begin{cases} \frac{C}{2\epsilon_k + \Delta} & \text{when } 0 < \epsilon_k < \Theta \\ 0 & \text{otherwise} \end{cases} \]
   where \( \Delta \) is the binding energy, \( \epsilon_k = \hbar^2 k^2 / 2m - \epsilon_F \), and \( C \) is a normalization constant, to get the result (valid for \( \Delta \ll \Theta \))
   \[ \xi \equiv < r^2 > \approx \frac{\hbar v_F}{\Delta}. \]
   Evaluate the coefficient \( c \). Note that \( k_F \xi \approx \epsilon_F / \Delta \), which is a large number, typically \( 10^4 \).

2. Fermi-Dirac Distribution
   As practice using creation and destruction operators, derive the Fermi-Dirac distribution in the following way. First we define a particular version of an average. Let \( < \hat{A} >_0 \) mean the canonical thermal average expectation of the operator \( \hat{A} \) in an equilibrium system defined by a Hamiltonian \( \mathcal{H}_0 \), that is
   \[ < \hat{A} >_0 = \frac{\text{tr} \hat{A} e^{-\beta(\mathcal{H}_0 - \mu \hat{N})}}{\text{tr} e^{-\beta(\mathcal{H}_0 - \mu \hat{N})}}. \]
   Let \( \mathcal{H}_0 \) be a single particle system \( \mathcal{H}_0 = \sum_i e_i c_i^\dagger c_i \) whose single-particle states are labeled by quantum numbers \( i = (\ldots) \) where the list includes the \( \hat{z} \) component of spin in some convenient coordinate system. The trace operation goes over the states \( |n_1, n_2, \ldots \rangle \) of \( \mathcal{H}_0 \). Show that \( < c_i^\dagger c_i >_0 = 0 \) (where states 1 and 2 are different, and that \( < c_1^\dagger c_1 > = f_1 \) (where \( f_1 \) is the Fermi-Dirac distribution.)
3. **Electron-electron interaction**

For the same system, evaluate $< \hat{V} >_0$, where

$$\hat{V} = \frac{1}{2} \sum_{a,b,c,d} <ab|vd|cd> c_{a\sigma}^+ c_{b\sigma} c_{c\sigma}^+ c_{d\sigma}^+.$$

That is, write an expression for $< \hat{V} >_0$ in terms of matrix elements $<ab|vd|cd>$ and other defined quantities such as single-particle energies $\epsilon_a$, single-particle equilibrium distributions $f_a$ etc. Write out the $\vec{r}$-space integral form of the required matrix elements. Explain the relation to Hartree-Fock theory (in general, this is not the full Hartree-Fock energy.) Notice the role of spin in the exchange term.